NOTE: FOR THE FOLLOWING PROBLEMS f AND g ARE FUNCTIONS AND c, α AND β ARE CONSTANTS

1 AN OPERATOR, $\hat{A}$, IS LINEAR IF IT SATISFIES BOTH:

$$\text{(i)} \quad \hat{A}(f+g) = \hat{A}f + \hat{A}g$$

AND (ii) $\hat{A}(cf) = c\hat{A}f$

a) GIVEN $\hat{A}f = \alpha f + \beta$, WE NOTE THAT $\hat{A}$ IS NOT LINEAR.

FOR (i) $\quad$ LHS $= \hat{A}(f+g) = \alpha(f+g)+\beta = \alpha f + \alpha g + \beta$

RHS $= \hat{A}f + \hat{A}g = \alpha f + \beta + \alpha g + \beta = \alpha f + \alpha g + 2\beta$

$\boxed{\text{SO LHS} \neq \text{RHS}}$

[NOTE: CHECKING (ii) IS UNNECESSARY AS $\hat{A}$ FAILS TO BE LINEAR FOR FAILING (i), BUT FOR COMPLETENESS WE CHECK (ii) ANYWAY]

FOR (ii) $\quad$ LHS $= \hat{A}(cf) = \alpha(cf) + \beta = c\alpha f + \beta$

RHS $= c\hat{A}f = c(\alpha f + \beta) = c\alpha f + c\beta$

$\boxed{\text{SO LHS} \neq \text{RHS}}$

b) GIVEN $\hat{A}f = \alpha \dfrac{d^2 f}{dx^2}$, WE NOTE THAT $\hat{A}$ IS LINEAR.

FOR (i) $\quad$ LHS $= \hat{A}(f+g) = \alpha \dfrac{d^2}{dx^2}(f+g) = \alpha \dfrac{d^2 f}{dx^2} + \alpha \dfrac{d^2 g}{dx^2}$

RHS $= \hat{A}f + \hat{A}g = \alpha \dfrac{d^2 f}{dx^2} + \alpha \dfrac{d^2 g}{dx^2}$

$\boxed{\text{SO LHS} = \text{RHS}}$

FOR (ii) $\quad$ LHS $= \hat{A}(cf) = \alpha \dfrac{d^2}{dx^2}(cf) = c\alpha \dfrac{d^2 f}{dx^2}$

RHS $= c\hat{A}f = c\alpha \dfrac{d^2 f}{dx^2}$

$\boxed{\text{SO LHS} = \text{RHS}}$

c) GIVEN $\hat{A}f = |f|$, WE NOTE THAT $\hat{A}$ IS NOT LINEAR

IT FAILS (i) SINCE $|f+g| \neq |f| + |g|$ IN GENERAL

AND IT FAILS (ii) SINCE $|cf| \neq c|f|$ FOR ALL c,

d) GIVEN $\hat{A}f = \dfrac{df^3}{dx}$, WE NOTE THAT $\hat{A}$ IS NOT LINEAR,

WHILE $\hat{A}$ SATISFIES (ii) IT FAILS CONDITION (i)

FOR (i)

$$\text{LHS} = \hat{A}(f+g) = \frac{d}{dx}(f+g)^3 = 3(f+g)^2(f'+g')$$

$$\text{RHS} = \hat{A}f + \hat{A}g = \frac{df^3}{dx} + \frac{dg^3}{dx} = 3f^2f' + 3g^2g'$$

SINCE THE LHS EXPANDS TO GIVE CROSS-TERMS THAT DO NOT APPEAR ON THE RHS

$$\boxed{\text{LHS} \neq \text{RHS}}$$

e) GIVEN $\hat{A}f = x^2 f$, WE NOTE THAT $\hat{A}$ IS LINEAR.

FOR (i)

$$\text{LHS} = \hat{A}(f+g) = x^2(f+g) = x^2 f + x^2 g$$

$$\text{RHS} = \hat{A}f + \hat{A}g = x^2 f + x^2 g$$

$$\boxed{\text{SO LHS} = \text{RHS}}$$

FOR (ii)

$$\text{LHS} = \hat{A}(cf) = x^2(cf) = cx^2 f$$

$$\text{RHS} = c\hat{A}f = cx^2 f$$

$$\boxed{\text{SO LHS} = \text{RHS}}$$

2. GIVEN TWO OPERATORS, $\hat{A}$ AND $\hat{B}$, WE SAY $\hat{A}$ & $\hat{B}$ COMMUTE IF $\hat{A}\hat{B}f = \hat{B}\hat{A}f$. (1)

ALTERNATIVELY, WE CAN DEFINE THE COMMUTATOR OPERATOR, $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$, AND CHECK THAT

$$[\hat{A},\hat{B}]f = 0. \quad (2)$$

a) GIVEN $\hat{A}f = \frac{df}{dx}$ AND $\hat{B}f = f^2$, WE NOTE THAT $\hat{A}$ AND $\hat{B}$ **DO NOT** COMMUTE.

CHECKING EXPRESSION (1) WE GET:

$$\begin{aligned} LHS &= \hat{A}\hat{B}f \\ &= \hat{A}(\hat{B}f) \\ &= \hat{A}f^2 \\ &= \frac{df^2}{dx} = 2ff' \end{aligned} \qquad \begin{aligned} RHS &= \hat{B}\hat{A}f \\ &= \hat{B}(\hat{A}f) \\ &= \hat{B}\frac{df}{dx} = \hat{B}f' \\ &= (f')^2 \end{aligned}$$

$\boxed{\text{SO } LHS \neq RHS}$

[NOTE: (1) AND (2) ARE EQUIVALENT, SO WE ONLY NEED TO CHECK ONE OF THEM → FOR PRACTICE CHECK (2)]

b) GIVEN $\hat{A}f = \frac{df}{dx}$ AND $\hat{B}f = \frac{d^2f}{dx^2}$, WE NOTE THAT $\hat{A}$ AND $\hat{B}$ **DO** COMMUTE.

LET US CHECK EXPRESSION (2) THIS TIME:

$$[\hat{A},\hat{B}]f = \hat{A}\hat{B}f - \hat{B}\hat{A}f = \hat{A}\left(\frac{d^2f}{dx^2}\right) - \hat{B}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right) - \frac{d^2}{dx^2}\left(\frac{df}{dx}\right)$$

$$= \frac{d^3f}{dx^3} - \frac{d^3f}{dx^3} = 0$$

SINCE $[\hat{A},\hat{B}] = 0$, $\hat{A}$ AND $\hat{B}$ COMMUTE

c) GIVEN $\hat{A}f = \alpha f + \beta$ AND $\hat{B}f = f^2$, WE NOTE THAT $\hat{A}$ AND $\hat{B}$ DO NOT COMMUTE

USING THE COMMUTATOR AGAIN, WE GET:

$$[\hat{A}, \hat{B}]f = \hat{A}\hat{B}f - \hat{B}\hat{A}f = \hat{A}f^2 - \hat{B}(\alpha f + \beta)$$
$$= (\alpha f^2 + \beta) - (\alpha f + \beta)^2$$
$$= \alpha f^2 + \beta - \alpha^2 f^2 - 2\alpha\beta f - \beta^2$$
$$= \alpha(1-\alpha)f^2 - 2\alpha\beta f + \beta(1-\beta)$$

SINCE $[\hat{A}, \hat{B}] \neq 0$, $\hat{A}$ + $\hat{B}$ DO NOT COMMUTE.

d) GIVEN $\hat{A}f = \frac{df}{dx}$ AND $\hat{B}f = f$, WE NOTE THAT $\hat{A}$ + $\hat{B}$ DO COMMUTE

LET US CHECK (1):

$$\text{LHS} = \hat{A}\hat{B}f = \hat{A}(\hat{B}f) = \hat{A}f = \frac{df}{dx}$$

$$\text{RHS} = \hat{B}\hat{A}f = \hat{B}(\hat{A}f) = \hat{B}\frac{df}{dx} = \frac{df}{dx}$$

$\boxed{\text{SO LHS = RHS}}$

3) FOR AN OPERATOR, $\hat{A}$, THE ASSOCIATED EIGENVALUE PROBLEM CAN BE WRITTEN AS:

$$\hat{A}f = \lambda f$$, WHERE f IS A FUNCTION AND λ IS A CONSTANT

f IS AN EIGENFUNCTION OF $\hat{A}$ AND λ IS THE EIGENVALUE ASSOCIATED WITH f.

IF $\hat{A}$ IS A LINEAR OPERATOR, THEN IF f IS AN EIGENFUNCTION OF $\hat{A}$, SO IS $g = c \cdot f$ FOR ANY CONSTANT c. [BY PROPERTY (ii) OF Q1]

a) GIVEN $\hat{A}f = \frac{df}{dx}$, THE EIGENVALUE PROBLEM WE WANT TO CONSIDER IS:

$$\frac{df}{dx} = \lambda f$$

WE CAN SOLVE THIS DIFFERENTIAL EQ'N BY SEPARATION OF VARIABLES.

$$\frac{df}{dx} = \lambda f$$

$$\frac{df}{f} = \lambda dx$$, INTEGRATE BOTH SIDES

$$\ln f = \lambda x + C$$

$$f = \exp\{\lambda x + c\}$$

$$\boxed{f = e^{c} e^{\lambda x} = K e^{\lambda x}}$$

SO THE EIGENFUNCTIONS OF $\hat{A}$ ARE GIVEN BY $f(x) = K e^{\lambda x}$ AND HAVE AN EIGENVALUE λ.

b) GIVEN $\hat{A}f = -\alpha \frac{d^2 f}{dx^2}$, THE EIGENVALUE PROBLEM WE WANT TO CONSIDER IS:

$$-\alpha \frac{d^2 f}{dx^2} = \lambda f$$

OR

$$\frac{d^2 f}{dx^2} = -\frac{\lambda}{\alpha} f = -\omega^2 f \quad \text{WHERE } \omega = \sqrt{\frac{\lambda}{\alpha}}$$

THE DIFFERENTIAL EQUATION $\frac{d^2 f}{dx^2} = -\omega^2 f$

IS WELL KNOWN AND HAS TWO SOLUTIONS

$$f_1(x) = A\sin(\omega t) \quad \text{AND} \quad f_2(x) = B\cos(\omega t)$$

THE GENERAL SOLUTION TO THIS DIFFERENTIAL EQUATION IS A COMBINATION OF THESE TWO SOLUTIONS, NAMELY:

$$F(x) = f_1(x) + f_2(x) = A\sin(\omega t) + B\cos(\omega t)$$

AND HAS AN ASSOCIATED EIGENVALUE

$$\lambda = \alpha\omega^2.$$

[NOTE: THIS IS A 2^{ND} ORDER ODE. AND AS SUCH SHOULD HAVE ONLY TWO SOLUTIONS WITH THE GENERAL SOLUTION BEING GIVEN BY A COMBINATION OF THOSE SOLUTIONS.

AS WITH ANY OTHER D.E., YOU CAN ALWAYS VERIFY THE SOLUTIONS BY SUBSTITUTING THEM BACK INTO THE D.E.]

[NOTE: THE SOLUTIONS TO THIS ODE CAN ALSO BE WRITTEN AS

$$g_1(x) = Ce^{i\omega x} \quad \text{AND} \quad g_2(x) = De^{-i\omega x}$$

WITH $G(x) = g_1(x) + g_2(x)$ AS THE GENERAL SOLUTION

BUT WE CAN SHOW THAT $F(x) = G(x)$ BY USING

$$e^{it} = \cos(t) + i\sin(t)$$ AND APPROPRIATE CHOICES

FOR THE CONSTANTS $C + D$.]

c) GIVEN $\hat{A} f = \alpha \dfrac{d^2 f}{dx^2}$, THE ASSOCIATED EIGENVALUE PROBLEM IS GIVEN BY:

$$\alpha \frac{d^2 f}{dx^2} = \lambda f \quad \text{OR} \quad \frac{d^2 f}{dx^2} = \omega^2 f \quad \text{WHERE } \omega = \sqrt{\frac{\lambda}{\alpha}}$$

THE ABOVE D.E. HAS TWO SOLUTIONS, GIVEN BY:

$f_1(x) = A e^{\omega x}$ AND $f_2(x) = B e^{-\omega x}$ AND A GENERAL SOLUTION $F(x) = A e^{\omega x} + B e^{-\omega x}$

WITH AN ASSOCIATED EIGENVALUE $\lambda = \alpha \omega^2$.

[NOTE: AGAIN, WE CAN RE-EXPRESS THE SOLUTIONS f_1 f_2 AND F IN TERMS OF ANOTHER SET OF FUNCTIONS

THIS TIME, USE THE HYPERBOLIC TRIG FUNCTIONS. IF WE TAKE

$$g_1(x) = C \sinh(\omega x) \quad \text{AND}$$
$$g_2(x) = D \cosh(\omega x)$$

THEN $G(x) = C \sinh(\omega x) + D \cosh(\omega x)$

IS THE GENERAL SOLUTION.

BUT $G(x) = F(x)$ IF WE USE THE RELATIONS

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \quad \text{AND} \quad \sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

AND APPROPRIATE CHOICES FOR C AND D].