Minimal model of a heat engine: Information theory approach

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We construct a generic model for a heat engine using information theory concepts, attributing irreversible energy dissipation to the information transmission channels. Using several forms for the channel capacity, classical and quantum, we demonstrate that our model recovers both the Carnot principle in the reversible limit, and the universal maximum power efficiency expression of nonreversible thermodynamics in the linear response regime. We expect the model to be very useful as a testbed for studying fundamental topics in thermodynamics, and for providing new insights into the relationship between information theory and actual thermal devices.

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I. INTRODUCTION

The Maxwell's demon puzzle, suggesting a violation of the second law of thermodynamics, has been exorcised using Landauer's memory erasure principle [1]. Treating the demon's intelligence as information has manifested a fundamental connection between information and physics, revealing the specific role of memory in terms of the second law of thermodynamics [2].

The goal of this paper is to suggest a minimal-universal model of heat engine (demon) using information theory concepts, and to employ it for analyzing the operation principles of finite-time thermodynamic machines [3]. From an information theoretic perspective, the engine described presents a physical implementation of information processing, manifesting the fundamental thermodynamic cost of computational operations. In our engine irreversible energy dissipation is attributed to information transfer within the demon's communication channels, compensating entropy decrease in the system, overwriting previous information. The model serves as a testbed for studying fundamental topics in thermodynamics. In particular, we examine the universality of the maximum power efficiency and its relationship to the Carnot efficiency.

A central topic in thermodynamics is the study of the efficiency of thermal engines, manifesting universal features. According to the Carnot result the efficiency (work output over heat input) of a cyclic thermal machine, operating between two heat baths of temperatures T_C and T_H ($T_C<T_H$), is at most

$$\eta_C = 1 - T_C/T_H. \tag{1}$$

The upper limit is obtained for engines that work reversibly. However, as reversible processes occur infinitesimally slowly, the power produced is zero. Operating away from equilibrium, a more practical question is the efficiency at *maximum power*, η_M, optimizing an engine cycle with respect to its *power* rather than efficiency. For a specific model, Curzon and Ahlborn (CA) derived the maximum power efficiency [4]

$$\eta_{CA} = 1 - \sqrt{T_C/T_H} = 1 - \sqrt{1-\eta_C} \approx \eta_C/2 + \eta_C^2/8 + \mathcal{O}(\eta_C^3), \tag{2}$$

relaying on the endoreversible approximation in which the sole source of irreversibility is due to heat transfer processes [3]. Does Eq. (2) represent a universal characteristic of finite-time thermodynamics? In the linear response regime it has been recently proven that the efficiency at maximum power is upper bounded by Eq. (2), which in this regime is exactly half of the Carnot efficiency [5]. The upper limit is reached for a specific class of "strongly coupled" models for which the energy flux is directly proportional to the work-generating flux. In contrast, in the nonlinear regime general results are missing [6,7].

In our model irreversible loss of energy occurs at the communication channels, transmitting classical information (entropy) from a heat source (T_H) to the engine (T_C). We consider both classical and quantum channels, encoding classical information in quantum states, and recover the Carnot efficiency for reversible operation, and the maximum power efficiency $\eta_C/2$ in the linear regime (small temperature differences). Using a generic form for the channel capacity we show that an agreement to higher terms in Eq. (2) can be obtained for a class of channels. Our study indicates on the fundamental status of the linear CA efficiency in irreversible thermodynamics. We find that if a model obeys the Carnot limit for a reversible process, it must also follow, for a nonreversible operation, the universal behavior, $\eta_M=\eta_C/2$ in the linear response regime, and vice versa. Moreover, the ratio of 1/2 between the two efficiencies holds even *below* the upper bounds, for imperfect systems.

II. MODEL

Our engine model (the right hand side of Fig. 1) is a physical realization of the Maxwell's demon, manifesting that entropy decrease in the heat source (left side of Fig. 1) should be compensated by entropy increasing processes in the demon (engine). The heat source is maintained at the temperature T_H, and the engine is kept at T_C. In each working cycle the engine absorbs the energy δE (e.g., by conduction or radiation) from the heat source in an isothermal re-

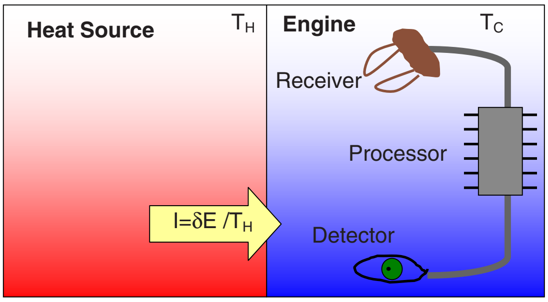

FIG. 1. (Color online) A scheme of the heat engine, see the text for details.

versible manner. The entropy loss of the heat source is thus $\Delta S_s=-\delta E/T_H$. Following Brillouin [8], Landauer [1], and others [2], this entropy change is identified as the amount of information I transmitted in the engine during each cycle

$$I=\delta E/T_H. \tag{3}$$

The engine itself consists four components: detector, communication channel, processor and receiver. Following Shanon's model for a communication system [9], in a working cycle I is encoded and sent through the engine's transmission channels to the processor which will decode it. Based on this information, the processor sends an instruction to properly set the internal states of the receiver, to accept the energy δE in a dissipationless manner. Since measurement of the heat source state and receiving its energy can be done (virtually) without energy consumption [2], this idealized engine apparently violates the second law, similarly to Szilard's toy engine [10]. One should thus look for an entropy increase of at least the same amount as in Eq. (3) somewhere in the demon, so as to compensate this apparent entropy decrease.

Szilard's demon has been exorcised by Landauer [1,2], considering the role of memory, demonstrating that dissipation due to the erasure of information *in the demon* compensates the entropy decrease in the heat source. In particular, in our model we attribute energy dissipation in the engine Q [see Eq. (4) below] to the transmission of information in the engine's channels and the resulting setup of the receiver, corresponding to Landauer's erasure principle [11]. We emphasize that heat transfer between the heat source and the engine does not take place through the information channels. Rather, only the *information* about the amount of heat that is going to be transferred is transmitted. Using this information, the engine then sets its internal states, so as to properly receive this energy at the receiver end.

As a specific physical realization one may consider a heat source comprised of a heat bath and a collection of particles with discrete energy levels, e.g., N two-level systems (TLS), characterized by the discrete states 0 and 1 of energies $\varepsilon_0=0$ and $\varepsilon_1=\varepsilon$, respectively. These TLS are assumed to be tightly attached to the heat bath, thus in thermal equilibrium the levels population follows the Boltzmann distribution $P_0=1/z$, $P_1=\exp(-\varepsilon/T_H)/z$; $z=(1+e^{-\varepsilon/T_H})$ is the partition function of the TLS, $k_B=1$. In each cycle the engine detects the state of the thermalized ensemble of TLS and absorbs its energy. For n-state particles the heat output from the heat source is identified by [12] $\delta E=N\Sigma_n\varepsilon_n dP_n$, or $\delta E=N\varepsilon(P_1-0)$ for a TLS brought to the ground state by the engine. The entropy gain of the engine is thus $I=N\varepsilon e^{-\varepsilon/T_H}/(zT_H)$.

III. ANALYSIS OF PERFORMANCE

The measure for a communication system is the channel capacity $I_p(S,N_0)$, describing the maximum number of bits that a system can communicate reliably per channel use. Here S and N_0 (a function of T_C) are the pulse and noise average energies, respectively [13]. In order to transfer the information I on the status of the heat source to the engine we need to split it into smaller pieces I_p, with the total energy cost

$$Q\equiv\frac{I}{I_p}S, \tag{4}$$

which cannot be refunded (else it contradicts the definition of maximum capacity), thus this is the heat dissipated in the engine during an operation cycle. The engine efficiency is given by the amount of available working energy over the invested energy. Using $W=\delta E-Q$ for the work attained per cycle, we obtain

$$\eta=\frac{W}{\delta E}=\frac{\delta E-Q}{\delta E}=1-\frac{S}{T_H I_p}. \tag{5}$$

We will optimize the engine cycle (i) with respect to its efficiency, minimizing Q, and (ii) with respect to its power $P=W/n$, W is the work output. Here $n=I/I_p$, the number of pulses that are needed to transfer the information I in a working cycle, is proportional to the period of the engine. Utilizing Eqs. (3) and (4) we get

$$P=T_H I_p-S. \tag{6}$$

Maximizing the power with respect to the pulse power S and substituting the optimal value S_M into Eq. (5), provides the maximum power efficiency

$$\eta_M=1-\frac{S_M}{T_H I_p(S_M,T_C)}. \tag{7}$$

We consider next classical and quantum channels, derive η for the two scenarios mentioned above, and manifest that they independently follow a universal behavior. One may also study the total entropy change in the process

$$\Delta S_{tot}=-\frac{\delta E}{T_H}+\frac{Q}{T_C}=I\left[(1-\eta)\frac{T_H}{T_C}-1\right]. \tag{8}$$

For a reversible process with the Carnot efficiency, $\eta=1-T_C/T_H$, the total entropy change is zero. When $\eta=0$, i.e., all the energy δE is dissipated to heat Q without any work generated, $\Delta S_{tot}=\delta E/T_C-\delta E/T_H$, the standard entropy increase in the process of heat transfer between two heat reservoirs.

IV. EXAMPLES

A. Gaussian channel

Suppose we send information over a classical continuous memoryless channel of bandwidth B (measured in Hertz), subjected to an additive white Gaussian noise with a power spectral density N_0. Assuming a signal power $\mathcal{S}$, and taking BN_0 as the total noise power in a bandwidth B, one obtains the Shanon-Hartley capacity $C=B\ln(1+\frac{\mathcal{S}}{BN_0})$ [9]. Defining $S=\mathcal{S}/B$ as the average energy carried by the pulse and $I_p=C/B$ as a dimensionless capacity [13], we get

$$I_p = \ln(1 + S/N_0). \tag{9}$$

In order to transfer the information I on the status of the subsystem into the engine we invest $Q=I\frac{S}{\ln(1+S/T_C)}$ [see Eq. (4)], identifying $N_0=T_C$ as the thermal noise power spectral density.

1. Carnot efficiency

The minimum value of the last relation is $Q=IT_C$, obtained when $S\rightarrow 0$, i.e., when the engine works reversibly with minimal energy consumption. As we invest Q and gain δE, the net average energy gain of the engine is

$$W = \delta E - IT_C = \delta E(1 - T_C/T_H), \tag{10}$$

resulting in the maximal efficiency $\eta_C=1-T_C/T_H$.

2. Maximum power efficiency

We calculate η_M using the Gaussian capacity [Eq. (9)]. The power [Eq. (6)] reduces to

$$P = T_H \ln(1 + S/T_C) - S. \tag{11}$$

Taking $\frac{\partial P}{\partial S}=0$ yields the energy per pulse maximizing the power, $S_M=T_H-T_C$. We substitute this result into Eq. (7) and obtain the efficiency at maximal power $\eta_M=1-S_M/[T_H\ln(1+S_M/T_C)]$. In terms of the Carnot efficiency, $\eta_C=\frac{S_M}{T_H}$, we find that

$$\frac{\eta_M}{\eta_C} = \frac{1}{\eta_C} + \frac{1}{\ln(1-\eta_C)} = \frac{1}{2} + \frac{1}{12}\eta_C + \frac{1}{24}\eta_C^2 + \mathcal{O}(\eta_C^3). \tag{12}$$

For $\eta_C\rightarrow 1$, η_M converges to 1. Note that the coefficient of the second (quadratic) term is smaller than the value recovered in several spatially-symmetric systems [14].

B. Bosonic channel

We next send the same *classical* information, yet in the form of quantum states, over a memoryless quantum channel which is a bosonic field (e.g., an electromagnetic radiation): The message information is encoded into modes of frequency ν and average photon number $n_B(\nu)$ [15,16]. The signal power is denoted by $\mathcal{S}$; noise power is $N=\pi T_C^2/12\hbar$ [17]. Under the constraint that the message-ensemble-averaged energy of the channel is fixed, the capacity of this channel is given by $C=\frac{\pi}{6\hbar}(T_e-T_C)$, where T_e, the (effective) temperature of the decoder, is defined through the relation $\mathcal{S}+\frac{\pi T_C^2}{12\hbar}=\frac{\pi T_e^2}{12\hbar}$, leading to $C=\frac{\pi T_C}{6\hbar}[(12\hbar\mathcal{S}/\pi T_C^2+1)^{1/2}-1]$. Redefining $I_p=C/B$ and $S=\mathcal{S}/B$; B denotes an effective bandwidth, we obtain

$$I_p = \frac{\pi T_C}{6B\hbar}\left[\left(\frac{12\hbar SB}{\pi T_C^2} + 1\right)^{1/2} - 1\right]. \tag{13}$$

Remarkably, this capacity is inherently linked to the thermal conductance κ of a ballistic quantum wire [17].

1. Carnot efficiency

We again minimize heat production in the channel $Q=IS/I_p$, using Eqs. (3) and (13). The minimum value of this expression is ($\lambda\equiv 6B\hbar/\pi$)

$$Q_{S\rightarrow 0} = \lim_{S\rightarrow 0}\frac{\delta E}{T_H}\frac{\lambda S/T_C}{(2\lambda S/T_C^2 + 1)^{1/2} - 1} = \delta E T_C/T_H, \tag{14}$$

retrieving the Carnot limit, $\eta_C=(\delta E-Q_{S\rightarrow 0})/\delta E=(1-T_C/T_H)$.

2. Maximum power efficiency

Maximizing P, we obtain $S_M=\frac{T_C^2}{2\lambda}[(\frac{T_H}{T_C})^2-1]$. We plug it into Eq. (7) and exactly resolve *only* the linear term

$$\eta_M/\eta_C = 1/2. \tag{15}$$

Interestingly, this result holds in both the high temperature limit, when the classical Shanon capacity is recovered $I_p=S/T_C$, as well as at low temperatures when quantum effects become important and the channel capacity approaches $I_p=\sqrt{\pi S/3\hbar B}$.

We proceed and explore a narrowband photon channel, $B\ll\nu$, where ν is the channel central frequency. In this case the noise power is given by $N=h\nu n_B(\nu,T_C)B$; $n_B(\nu,T)=[e^{h\nu/T}-1]^{-1}$ is the Bose-Einstein distribution. Based on the assumption that photon noise is additive, the (dimensionless) capacity $I_p=C/B$ fulfills [9]

$$I_p = \ln\left[1 + \frac{S}{h\nu}(1 - e^{-h\nu/T_C})\right] + [S/h\nu + n_B(\nu,T_C)] \times\ln\left[1 + \frac{h\nu}{S + h\nu n_B(\nu,T_C)}\right] - n_B(\nu,T_C)h\nu/T_C. \tag{16}$$

Using this expression, the Carnot limit can be analytically obtained. Since analytical results are cumbersome, we acquire S_M numerically, seeking the intersect of $1/T_H$ with $\partial I_p(S)/\partial S$, see Eq. (6). The results, displayed in Fig. 2, clearly demonstrate that η_M is bounded by η_{CA} from above, and that $\eta_M=\eta_C/2$ in the lowest order of $\Delta T/T$. Interestingly, the results are almost independent of the channel central frequency ν. The inset displays channel capacities for the three models considered: Gaussian, wideband and narrowband bosonic channels. Figure 3 further shows the maximum power efficiency η_M for the classical and quantum wideband channels, as a function of η_C, compared to the CA and the Carnot efficiencies. We find that the CA value is an upper bound for η_M.

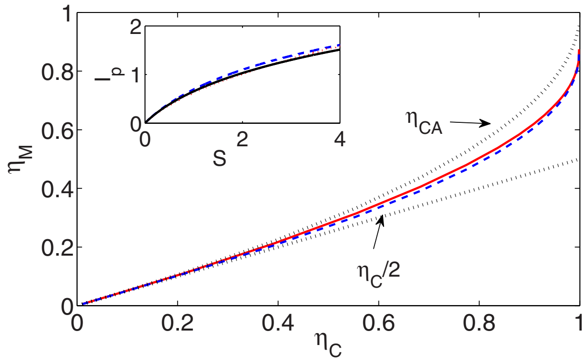

FIG. 2. (Color online) Maximum power efficiency for a heat engine of a narrowband bosonic channel with $\nu=0.2$ (full) and $\nu=2$ (dashed). $T_C=1$ was kept fixed. The CA efficiency and half of the Carnot limit are shown as dotted lines. Inset: Gaussian (dashed line), wideband bosonic (dotted line) and narrowband bosonic with $\nu=2$ (full line) capacities, $T_C=1$.

C. Generic channel

Motivated by the classical-Gaussian channel and the quantum models, we consider next a general form for a channel capacity,

$$I_p = \alpha x + \beta x^2 + \gamma x^3 + \mathcal{O}(x^4). \tag{17}$$

Here $x=S/T_C$ is the signal power over noise, and the parameters α, β, and γ may depend on T_C. These coefficients are constricted so as to produce positive capacity, $I_p \geq 0$. Assuming I_p is a concave function, valid for the cases investigated above, the inequality $3\gamma S < -\beta T_C$ must be satisfied. Since $S>0$ and $\gamma>0$ [so as $I_p(S \geqslant 0)>0$], we conclude that β must be negative. In particular, for the Gaussian channel [Eq. (9)] a power expansion gives $\alpha=1$, $\beta=-1/2$ and $\gamma=1/3$, while for the bosonic wideband quantum channel we get $\alpha=1$, $\beta=-3\hbar B/\pi T_C$ and $\gamma=18\hbar^2 B^2/\pi^2 T_C^2$. Using the generic form Eq. (17), the maximal efficiency is obtained by minimizing heat production, $Q=IS/I_p$, resulting in

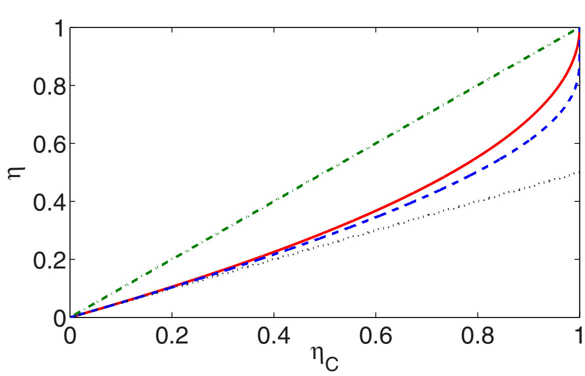

FIG. 3. (Color online) Maximum power efficiency using different models for the information channel: Gaussian [Eq. (9)] (dashed line), wideband Bosonic [Eq. (13)] (dotted), compared to the Curzon-Ahlborn efficiency (full) and the Carnot efficiency (dashed-dotted line).

$$Q(S \to 0) = IT_C/\alpha; \quad \eta = \tilde{\eta}_C/\alpha, \tag{18}$$

where $\tilde{\eta}_C = \eta_C + \alpha - 1$. Thus, in order to reach the Carnot limit one must require that $\alpha=1$. Next, still using the generic channel capacity form [Eq. (17)] we maximize P and obtain the pulse power $S_M = \frac{-\beta T_C}{3\gamma}[1-(1-\frac{3\gamma\tilde{\eta}_C}{\beta^2})^{1/2}]$. Plugging this value into Eq. (7) leads to

$$\eta_M = \frac{\tilde{\eta}_C}{2\alpha} + \frac{\tilde{\eta}_C^2}{4}\left(\frac{1}{\alpha^2} - \frac{\gamma}{2\beta^2\alpha}\right) + \mathcal{O}(\tilde{\eta}_C^3). \tag{19}$$

Imposing $\alpha=1$ results in

$$\eta_M/\eta_C = \frac{1}{2} + \frac{\eta_C}{4}(1-\gamma/2\beta^2) + \mathcal{O}(\eta_C^2). \tag{20}$$

The following important observations can be made: (i) The Carnot limit $\eta_C=(T_H-T_C)/T_H$ and the efficiency at maximum power $\eta_M=\eta_C/2$ (at first order in η_C) are nonsubordinate elemental results in thermodynamics: Demanding that $\alpha=1$ for channels working at maximal capacity, independently produces both the Carnot efficiency, and the universal linear coefficient in Eq. (20). (ii) A universal relation between the two efficiencies stands still even *below* the upper bound: assuming that $\alpha<1$ accounts for loss mechanisms in the channel, operating below its full capacity, the maximal efficiency of the engine is given by $\tilde{\eta}/\alpha$, while the maximal power efficiency (in linear response) is exactly half this value, $\tilde{\eta}/2\alpha$. (iii) The first (linear) term in Eq. (19) is independent of the details of the channel model, β and γ. (iv) Finally, the (lack of) universality in the quadratic term of η_M has been a source of debates [14]. Our analysis exposes that this coefficient depends on the details of the channel, i.e., on the relations between the nonlinear terms. In particular, if $\gamma=2\beta^2$ the quadratic term diminishes, in agreement with the quantum (wideband) bosonic channel. When $4\beta^2=3\gamma$, it reduces to $\eta_C^2/12$, as in the classical-Gaussian channel. For $\gamma=\beta^2$ the quadratic term becomes $\eta_C^2/8$, recovering the CA result [Eq. (2)] and other symmetric models [14].

V. SUMMARY

We have designed here a minimal model of a heat engine (or a Maxwell's demon), attributing energy dissipation within the machine to irreversible loss of energy within the engine communication channels. The significance of the model is that it relies on abstract information theory concepts and thus is independent on specific physical realizations. Analyzing both classical and quantum information channels we have demonstrated that our model satisfies the Carnot limit for reversible operation mode. Independently, it leads to the universal linear term in the maximum power efficiency, for a nonreversible process, exhibiting the existence of fundamental laws for finite-time thermodynamic processes. While the universality of the maximal power efficiency in the linear response regime has been explained based on the Onsager's symmetry [5], our model establishes this universality developing a distinct, information based picture.

We expect this prototype model to be very useful for studying other basic concepts in thermodynamics. For example, the engine described here could serve as a cooling device, or the performance could be analyzed for a subsystem away from the Boltzmann distribution. One could also advance the model beyond the strong coupling limit, $W \propto \delta E$ [18], or consider other sources of dissipation. Finally, it is of great interest to further include quantum effects, considering transmission of quantum information.

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