



SPIRAL WAVES IN MEDIA WITH COMPLEX-EXCITABLE DYNAMICS

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The structure of spiral waves is investigated in super-excitable reaction–diffusion systems where the local dynamics exhibits multilooped phase-space trajectories. It is shown that such systems support stable spiral waves with broken rotational symmetry and complex temporal dynamics. The main structural features of such waves, synchronization defect lines, are demonstrated to be similar to those of spiral waves in systems with complex-oscillatory dynamics.

1. Introduction

Studies of spatially-distributed active media have demonstrated the ubiquity of self-organized spatiotemporal patterns, in particular spiral waves, in various physical or biological systems such as Belousov–Zhabotinsky (BZ) reaction (see for review [Zhabotinsky, 1991], catalytic surfaces [Sales *et al.*, 1982], cardiac muscle (see e.g. [Winfree, 1991; Tyson & Keener, 1987] and colonies of the amoebae *Dictyostelium discoideum* [Devreotes, 1989].

Most research has been devoted to the study of simple oscillatory or excitable systems. Recently it was shown that reactive media with complex periodic and chaotic oscillations are capable of supporting spiral waves [Klevecz *et al.*, 1991; Goryachev & Kapral, 1996a] with a variety of distinctive features, absent in simple oscillatory systems [Goryachev & Kapral, 1996b; Goryachev *et al.*, 1998]. The rotational symmetry of spiral waves in period-doubled media is broken by synchronization line defects where the phase of the local oscillation changes by multiples of 2π . It was conjectured that spiral waves with broken rotational symmetry could also be observed in super-excitable systems where the phase-space trajectory after excitation follows a multilooped path of relaxation to the stable fixed point [Chialvo *et al.*, 1990]. Broken spirals with a

clearly visible synchronization defect line emanating from the spiral core were observed under special three-dimensional conditions in the BZ reactive medium [Yoneyama *et al.*, 1995]. The nature of spiral waves in complex-excitable media is nevertheless largely unexplored.

In this paper we show that spiral waves with broken rotational symmetry exist in a prototypical super-excitable system and demonstrate that the topological properties of line defects, described earlier for oscillatory media, also hold for excitable systems.

We consider a spatially-distributed system whose dynamics is governed by a pair of reaction–diffusion equations of the form

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{1}{\varepsilon} u(1-u) \left(u - \frac{v+b}{a} - f(v) \right) + D\nabla^2 u, \\ \frac{\partial v}{\partial t} &= u - v,\end{aligned}\tag{1}$$

where v is a nondiffusive variable and both u and v are functions of time and space. This model with $f(v) \equiv 0$ was studied in [Barkley, 1991] as a simplified version of the FitzHugh–Nagumo model which serves as a prototype of an excitable system described by two variables. The excitable dynamics of system (1) consists of two fast and two slow stages.

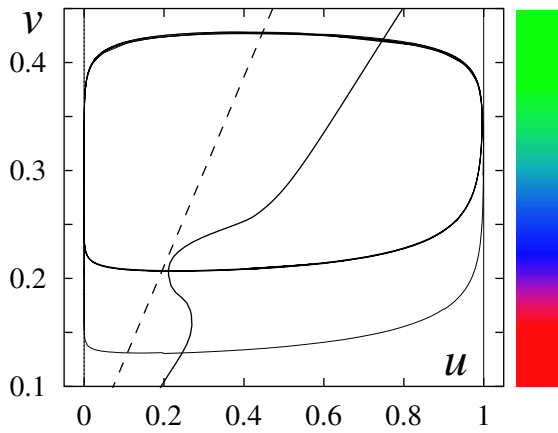


Fig. 1. Phase plane (u, v) plot of a $\mathbf{1}^2$ type trajectory calculated at a point in the medium shown in Fig. 4. Two stable branches ($u = 0$, $u = 1$) and one unstable branch of the nullcline $\dot{u} = 0$ are shown as solid lines and the nullcline $\dot{v} = 0$ by a dashed line.

If displaced from the stable fixed point $u = 0$, $v = 0$ to the right of the unstable branch of the nullcline $\dot{u} = 0$, it quickly reaches upper stable branch $u = 1$. It follows this branch until v reaches sufficiently large values and then jumps to the lower stable branch $u = 0$, along which it slowly relaxes to the stationary state. To add a super-excitability to (1) a modification of the unstable branch of the nullcline $\dot{u} = 0$ was proposed in [Giaquinta *et al.*, 1994] as

$$f(v) = \alpha \exp\left(-\frac{(v - v_0)^2}{\sigma^2}\right). \quad (2)$$

The introduction of (2) changes the shape of the unstable branch of $\dot{u} = 0$ so that for suitably chosen parameters α , σ and v_0 it nearly touches the nullcline $\dot{v} = 0$ (see Fig. 1). As a result, if another excitation is applied to the system (1) before it has reached the stationary state, it may execute second, smaller excitable loop before it finally reaches the stable state.

The spatiotemporal dynamics of a one-dimensional array of such elements forced by an external pacemaker with varying period was studied in [Giaquinta *et al.*, 1994]. When the period of forcing T_f is larger than a certain internal period of the system T_0 , the response is a train of waves corresponding to the large relaxation loop ($\mathbf{1}^0$). If $T_f < T_0$ the system develops wavetrains with low amplitude corresponding to the small relaxation loop ($\mathbf{0}^1$). Nontrivial behavior is observed when T_f is only slightly smaller than T_0 . In this case the system shows mixed-mode waveforms ($\mathbf{1}^n$)

consisting of one large and n small waves. Response of this type is an example of the complex-excitable dynamics targeted in our studies. Instead of an external pacemaker we use a spiral wave as a self-sustained source of excitation in the medium.

Spiral waves were initiated in a two-dimensional square domain with no-flux boundary conditions. While α and σ were used as bifurcation parameters, other parameters were fixed at ($\varepsilon = 0.005$, $a = 0.6$, $b = 0.03$, $v_0 = 0.2$). As in [Giaquinta *et al.*, 1994] complex-excitable dynamics was found in the parameter region between domains of large-amplitude $\mathbf{1}^0$ (small α and σ) and small-amplitude $\mathbf{0}^1$ (large α and σ) waves. In this region the medium supports mostly aperiodic, stable spiral waves lacking rotational symmetry. Figure 2 shows such a spiral wave for $\alpha = 0.15$, $\sigma^2 = 0.001$ at two time instances. Note that the wavelength of a large-amplitude wave is larger than that of a small-amplitude wave and, thus, the shape of the spiral is distorted. The concentration time-series $v(\mathbf{r}, t)$ at different locations in the medium (cf. Fig. 3) show aperiodic concatenations of $\mathbf{1}^1$ and $\mathbf{1}^2$ waveforms, while trivial patterns $\mathbf{1}^0$ and $\mathbf{0}^1$ are completely absent.

As the domain of complex-excitable dynamics is traversed from $\mathbf{1}^0$ to $\mathbf{0}^1$ the contribution of the waveforms $\mathbf{1}^n$ with $n > 0$ steadily grows, as well as the number n of the small-amplitude loops. Thus, the transition from large- to small-amplitude spiral waves occurs gradually through a succession of irregular spiral patterns with a progressively growing contribution of low-amplitude waves.

Although irregular patterns are found in most of the complex-excitable domain, pure period-3 dynamics was found in a subdomain of this region. Figure 4 shows the spatial structure of a spiral wave in this parameter region. The observation of the spiral wave dynamics for long time periods shows that the entire concentration field slowly rotates around the spiral core with a constant angular velocity ω . The period-3 dynamics is manifested in a coordinate frame centered at the spiral core and rotating with velocity ω . Indeed, it takes three rotations of the spiral for the concentration field to return to itself.

Figure 1 shows a phase portrait of the dynamics at a nonspecial location in the medium calculated in the rotating frame. Before closing onto itself the phase-space trajectory executes two small loops and one large loop, corresponding to a pure $\mathbf{1}^2$ dynamics. Consider a polar coordinate frame (ρ, ϕ) in the (u, v) phase plane with origin at an

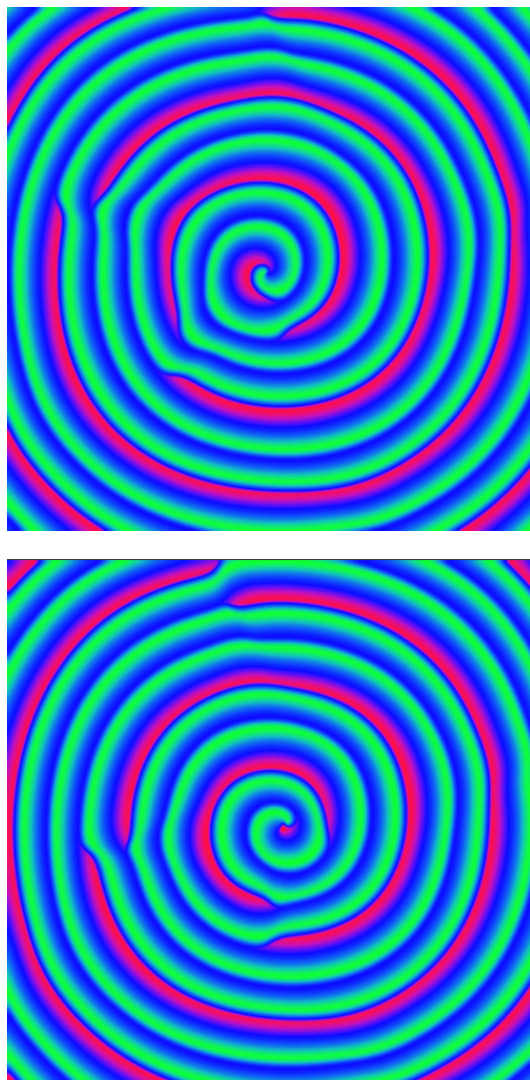


Fig. 2. An irregular spiral wave with complex excitable dynamics is shown at two different times in the upper and lower panels, respectively. The concentration field $v(\mathbf{r}, t_0)$ is color-coded as shown in Fig. 1.

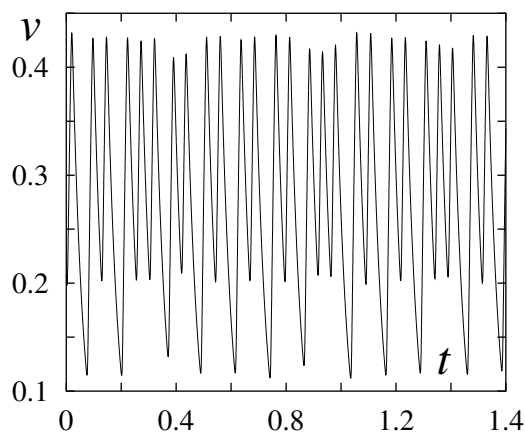


Fig. 3. Time-series $v(\mathbf{r}, t)$ calculated at a point in the medium shown in Fig. 2.

arbitrary point internal to both the small and large loops. During one full period of the dynamics the phase variable ϕ changes by 6π . The calculation of the phase at every point in the medium at a time t_0 gives an instantaneous snapshot $\phi(\mathbf{r}, t_0)$ of the time-dependent phase field $\phi(\mathbf{r}, t)$.

Consider a closed contour Γ that surrounds the spiral core. The phase increment $\Delta_{\Gamma}\phi = \oint_{\Gamma} \nabla\phi(\mathbf{r}, t_0)d\mathbf{l}$ along Γ will be equal to a multiple of the full 6π period of the dynamics. From the topological theory of point defects [Mermin, 1979] in simple oscillatory or excitable media it follows that $\Delta_{\Gamma}\phi$ is invariant and for the one-armed spiral waves in this study takes the values $\pm 2\pi$. In complex-periodic media this contradiction is resolved by the existence of synchronization defect lines emanating from the core [Goryachev & Kapral, 1996b; Goryachev *et al.*, 1998]. The phase of the local oscillation experiences jumps equal to multiples of 2π when such line defects are crossed. Any contour Γ encircling the spiral core intersects these lines so that the total phase increment $\Delta_{\Gamma}\phi$ is obtained from the integration of $\nabla\phi(\mathbf{r}, t_0)$ along Γ yielding $\pm 2\pi$ plus phase jumps at the intersections of Γ with the synchronization defect lines. The sum of both contributions yields the full period phase increment of the local dynamics. It was predicted [Goryachev & Kapral, 1996; Goryachev *et al.*, 1998] that one should be able to find phenomena analogous to synchronization defect lines in media with complex-excitable dynamics.

For the specific case of period-3 complex-excitable dynamics where the total phase increment is 6π , one might expect to find one line defect where the phase jumps by 4π or two defect lines where the phase jumps by 2π on crossing each line. In Fig. 4 one sees two lines emanating from the spiral core at an angle of $\sim 180^\circ$. Investigation of the change in phase of the local dynamics across these lines shows that they are indeed 2π synchronization defect lines exhibiting the loop exchange phenomenon described earlier for complex-oscillatory media.

Figure 5 shows the $v(\mathbf{r}, t)$ time-series at four neighboring locations in the medium along a path traversing one of the defect lines. Panel **a** is a plot of the normal $\mathbf{1}^2$ dynamics seen on one side of the defect line. Every third minimum is lower than the two preceding minima and corresponds to the larger relaxation loop in the phase-space plot. As one approaches the defect line, the large-amplitude loop shrinks while both small-amplitude loops grow [Fig. 5(b)]. Then, one small-amplitude loop

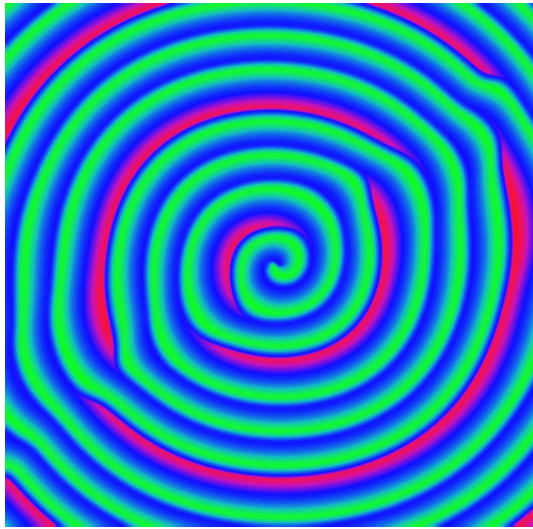


Fig. 4. Spiral wave with period-3 dynamics. Color-coding is the same as in Fig. 2.

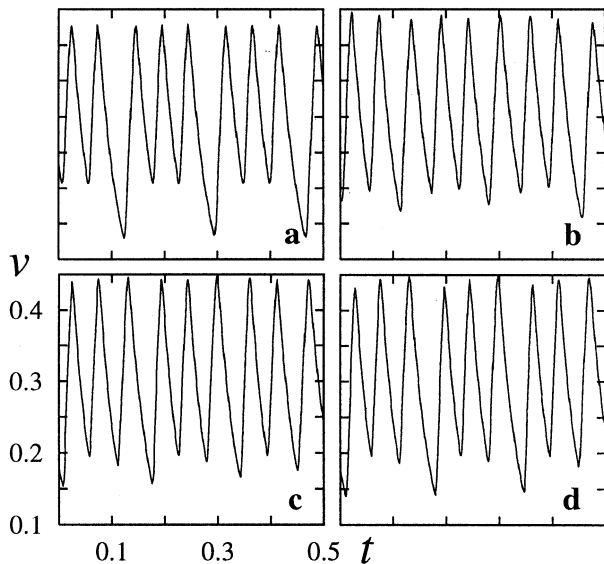


Fig. 5. Change in character of local dynamics across one of the defect lines seen in Fig. 4. See text for details.

begins to grow faster than the other and at some point becomes larger than the still shrinking large-amplitude loop [Fig. 5(c)]. This loop exchange process continues until the new large loop attains a size equal to that of a large-amplitude loop and the other two loops shrink to the size of the small-amplitude loop. The local dynamics on opposite sides of the defect line [compare Figs. 5(a) and 5(c)] experiences a 2π phase shift. Thus, the total phase increment $\Delta_{\Gamma}\phi = 2\pi$ resulting from the integration of $\nabla\phi(\mathbf{r}, t_0)$ along Γ , excluding its intersections with defect lines, plus the two 2π phase jumps at

the intersection points gives the expected 6π phase increment.

Our results show that the structure of complex-periodic spiral waves is governed by general topological principles independent of whether the dynamics is excitable or oscillatory. This fact allows one to extend the predictions inferred from the studies of complex-oscillatory systems to systems with super-excitable dynamics. The formation of complex-periodic spiral waves in such systems might play a role in the development of some pathological conditions in the heart. Indeed, mixed-mode electrical activity with alternating large and small amplitude maxima, so-called alternans, is typically observed as a symptom of tachycardia [Pulignano *et al.*, 1990].

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