

# The hydrodynamics of inelastic granular systems

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The hydrodynamic equations for a system of inelastic granular particles are derived from first principles of statistical mechanical theory by applying projection operator techniques. An effective Liouvillian operator for the granular distribution function is derived by exploiting the fact that each granular particle has many interacting internal degrees of freedom which remain at equilibrium at a temperature  $T$  and provide a sink for the translational relative momenta of the inelastic granular system. The nonlinear hydrodynamic equations for the granular system are obtained following projection operator techniques developed by Levine and Oppenheim. The resulting equations are similar to the ordinary hydrodynamic equations but contain additional terms due to the fact that translational energy is not conserved in collisions between the granular particles. The solutions to the linearized equations are also analyzed in different regimes comparing the additional terms due to the inelasticity of collisions with the magnitude of the gradients of the system.

## 1. Introduction

In recent years, there has been an intense study of the transport properties of granular fluid systems composed of sand or glass beads. Most of these studies have been semi-phenomenological and involve the concept of the coefficient of restitution, which is a measure of the inelasticity of collisions among the granular particles composing the fluid. As examples of these studies, we mention the analytical treatments of Jenkins et al. [1] and the simulations by Walton et al. [2]. The inelasticity is essential for the treatment of these systems and for comparisons with experimental data [3].

The aim of this paper is to derive the nonlinear hydrodynamic equations for a system of  $N$  particles in which the collisions between particles are inelastic. We expect the equations to apply to granular systems since energy flows unidirectionally from the translational degrees of freedom into internal modes of the particles in the collisions between granular particles. Each granular particle contains many molecules and internal modes which can be modelled as a bath with an internal temperature,  $T$ , which essentially remains in equilib-

rium throughout collisions. Note that when the translational temperature is  $T$ , the center of mass velocities of the particles are essentially zero. Using ideas developed to describe Brownian motion dynamics [4], in section 2 we obtain a generalized Fokker–Planck equation for the distribution function of the translational degrees of freedom of the granular system and thereby derive an effective Liouvillian operator which determines the time evolution of the distribution function. We also derive a generalized Smoluchowski equation for the spatial distribution function for the center of mass coordinates of the particles,  $R(\mathbf{r}^N, t)$ . From the Smoluchowski equation, the equation of motion for the radial distribution function  $R^{(2)}(\mathbf{r}, t)$  is obtained, and the stationary form of the radial distribution function for the granular flow system is examined. In section 3, we derive the nonlinear hydrodynamic equations for the granular system by generalizing a projection operator technique employed by Levine and Oppenheim [5] in their work on nonlinear transport properties. The linearized form of these new hydrodynamic equations are presented and analyzed in section 4 in various regimes reflecting the degree of inelasticity of each collision between granular particles. The solutions are contrasted with those for ordinary hydrodynamic systems in which the particles interact elastically. The final results are summarized in section 5 along with a discussion of the possibility of generalizing them.

## 2. The model and generalized Fokker–Planck equations

The granular system consists of  $N$  identical spherical particles whose center of mass coordinates and momenta are denoted by  $\mathbf{r}^N$  and  $\mathbf{p}^N$  respectively, where  $\mathbf{r}^N$  and  $\mathbf{p}^N$  are  $3N$ -dimensional vectors with components  $r_j$  and  $p_j$  with  $j = 1, \dots, N$ . Each of the  $N$  granular particles contains many internal modes. We will denote the internal coordinates of particle  $j$  by the vector  $\xi_j$ , and the internal momenta of particle  $j$  by  $\pi_j$ . We assume that the number of internal momenta and coordinates of each particle is sufficiently large and the interactions among them sufficiently strong that their distribution function is of equilibrium form. We denote the phase point for the translational degrees of freedom by  $X_t \equiv (\mathbf{r}^N, \mathbf{p}^N)$  and the phase point for the internal degrees of freedom by  $X_i \equiv (\xi^N, \pi^N)$ .

The Hamiltonian for the granular flow system can be written as

$$H(X_t, X_i) = H_t(X_t) + H_i(X_i) + \phi(X_t, X_i), \quad (2.1)$$

where  $H_t$  is the Hamiltonian for the isolated translational degrees of freedom,  $H_i$  is the Hamiltonian for the isolated internal degrees of freedom and  $\phi$

describes the interactions between the translational and internal degrees of freedom. The Hamiltonian  $H_t$  has the form

$$H_t(X_t) = \frac{\mathbf{p}^N \cdot \mathbf{p}^N}{2m} + U(\mathbf{r}^N), \tag{2.2}$$

where  $m$  is the mass of the granular particle and  $U$  can be written as a sum of two body short-ranged potentials,

$$U(\mathbf{r}^N) = \sum_{i=1}^N \sum_{j>i} U(r_{ij}), \tag{2.3}$$

where  $r_{ij} \equiv |\mathbf{r}_i - \mathbf{r}_j|$ . The Hamiltonian  $H_i$  is of the form

$$H_i = \sum_{j=1}^N \left( \frac{\boldsymbol{\pi}_j \cdot \boldsymbol{\pi}_j}{2\mu} + V(\boldsymbol{\xi}_j) \right). \tag{2.4}$$

The interaction term  $\phi$  can be written as a sum of two body terms, each of which depends on the distance between the center of mass of a pair of particles and on the internal coordinates of that pair,

$$\phi(X_t, X_i) = \sum_{i=1}^N \sum_{j>1} \phi(r_{ij}, \boldsymbol{\xi}_i, \boldsymbol{\xi}_j). \tag{2.5}$$

We assume that  $\phi$  is short-ranged and decays to zero for  $r_{ij} > \sigma$ , where  $\sigma$  represents the range of interaction, which we expect to be on the order of the diameter of a particle.

The Liouvillian operator for the system can be written as

$$L = L_t + L_i + L_\phi, \tag{2.6}$$

where

$$L_t = - \frac{\mathbf{p}^N}{m} \cdot \nabla_{\mathbf{r}^N} + \nabla_{\mathbf{r}^N} U \cdot \nabla_{\mathbf{p}^N}, \quad L_i = - \frac{\boldsymbol{\pi}^N}{\mu} \cdot \nabla_{\boldsymbol{\xi}^N} + \nabla_{\boldsymbol{\xi}^N} V \cdot \nabla_{\boldsymbol{\pi}^N} \tag{2.7}$$

and

$$L_\phi = \nabla_{\mathbf{r}^N} \phi \cdot \nabla_{\mathbf{p}^N} + \nabla_{\boldsymbol{\xi}^N} \phi \cdot \nabla_{\boldsymbol{\pi}^N}. \tag{2.8}$$

The Liouville operator  $L$  determines the dynamics of the total distribution function  $\rho(X_t, X_i, t)$  of the granular flow system,

$$\dot{\rho}(t) = L\rho(t). \tag{2.9}$$

We shall use a projection operator technique [6] to obtain an equation of motion for the reduced distribution function of the translational degrees of freedom  $W(X_t, t)$ , where

$$W(X_t, t) = \int dX_i \rho(X_t, X_i, t). \quad (2.10)$$

At total equilibrium, the distribution functions are

$$\rho_e = \frac{e^{-\beta H_t} e^{-\beta H_i} e^{-\beta \phi}}{q} \quad (2.11)$$

and

$$W_e = \frac{e^{-\beta H_t} \int dX_i e^{-\beta H_i} e^{-\beta \phi}}{q}, \quad (2.12)$$

where

$$q \equiv \int dX_t dX_i e^{-\beta H_t} e^{-\beta H_i} e^{-\beta \phi},$$

$\beta \equiv (k_B T)^{-1}$ , and  $T$  is the equilibrium temperature of the overall system. It will be useful to introduce the equilibrium distribution function for the isolated internal degrees of freedom,

$$\rho_i(X_i, t) \equiv \frac{e^{-\beta H_i}}{\int dX_i e^{-\beta H_i}}, \quad (2.13)$$

and define the potential of mean force,  $\omega(\mathbf{r}^N)$ , by

$$e^{-\beta \omega(\mathbf{r}^N)} \equiv \int dX_i \rho_i e^{-\beta \phi} \equiv \langle e^{-\beta \phi} \rangle_0. \quad (2.14)$$

Finally, we define  $\tilde{\rho}$  to be the conditional distribution function for the internal degrees of freedom in the presence of fixed translational degrees of freedom,

$$\tilde{\rho} \equiv \frac{\rho_e}{W_e} = \rho_i e^{-\beta \phi} e^{\beta \omega}. \quad (2.15)$$

Note that  $\tilde{\rho}$  is normalized in the sense that  $\int dX_i \tilde{\rho} = 1$ . We define the projection operator  $\mathcal{P}$  acting upon an arbitrary dynamical variable of the system  $B$  by the relation

$$\mathcal{P}B = \tilde{\rho} \int dX_i B . \tag{2.16}$$

By defining

$$y(t) \equiv \mathcal{P}\rho(t) = \tilde{\rho}W(t) \tag{2.17}$$

and

$$z(t) \equiv (1 - \mathcal{P})\rho(t) \equiv \mathcal{Q}\rho(t) , \tag{2.18}$$

it follows from eqs. (2.9) and (2.10) and the facts that

$$\int dX_i L_i B = 0 , \quad \int dX_i L_t B = L_t \int dX_i B$$

$$\int dX_i L_\phi B = \nabla_{p^n} \cdot \int dX_i \nabla_{r^N} \phi B ,$$

that

$$\dot{W}(t) = \int dX_i L[y(t) + z(T)]$$

$$= L_t W(t) + \int dX_i \tilde{\rho} \nabla_{r^N} \phi \cdot \nabla_{p^N} W(t) + \nabla_{p^N} \cdot \int dX_i \nabla_{r^N} \phi z(t) . \tag{2.19}$$

The equation of motion for  $z(t)$  is given by

$$\dot{z}(t) = \mathcal{Q}Lz(t) + \mathcal{Q}Ly(t) , \tag{2.20}$$

which has the formal solution

$$z(t) = e^{\mathcal{Q}L t} z(0) + \int_0^t d\tau e^{\mathcal{Q}L \tau} \mathcal{Q}Ly(t - \tau) . \tag{2.21}$$

Since

$$Ly(t) = (L\tilde{\rho})W(t) + \tilde{\rho}(L_t + \nabla_{r^N} \phi \cdot \nabla_{p^N})W(t)$$

and

$$L\tilde{\rho} = -\tilde{\rho}(L_t + \nabla_{r^N} \phi \cdot \nabla_{p^N}) \ln W_e ,$$

we find that

$$\mathcal{L}Ly(t) = \tilde{\rho} \left( \widehat{\nabla_{r^N} \phi} \cdot \nabla_{p^N} + \widehat{\nabla_{p^N} \phi} \cdot \frac{\beta p^N}{m} \right) W(t), \quad (2.22)$$

where

$$\widehat{\nabla_{r^N} \phi} \equiv \nabla_{r^N} \phi - \langle \nabla_{r^N} \phi \rangle_i = \nabla_{r^N}(\phi - \omega)$$

and  $\langle B \rangle_i \equiv \int dX_i \tilde{\rho} B$ . Substitution of the expressions for  $z(t)$  and  $\mathcal{L}Ly(t)$  in eqs. (2.21) and (2.22) into eq. (2.19) yields the exact equation

$$\begin{aligned} \dot{W}(X_i, t) = & \left( -\frac{p^N}{m} \cdot \nabla_{r^N} + \nabla_{r^N}(U + \omega) \cdot \nabla_{p^N} \right) W(t) \\ & + \nabla_{p^N} \cdot \int_0^t d\tau \int dX_i \nabla_{r^N} \phi e^{\mathcal{L}\tau} \tilde{\rho} \\ & \times \left( \widehat{\nabla_{r^N} \phi} \cdot \nabla_{p^N} + \widehat{\nabla_{p^N} \phi} \cdot \frac{\beta p^N}{m} \right) W(t - \tau) \\ & + \nabla_{p^N} \cdot \int dX_i \nabla_{r^N} \phi e^{\mathcal{L}t} z(0). \end{aligned}$$

In order to make the equation of motion for the reduced distribution function  $W(t)$  tractable, it is useful to exploit the difference between the mass of the granular particle and the effective mass  $\mu$  of the internal modes. We therefore define

$$\epsilon \equiv \left( \frac{\mu}{m} \right)^{1/2}, \quad (2.23)$$

which we expect to be very small ( $\epsilon \ll 1$ ) since the granular particle is massive. It is then convenient to define the reduced momentum  $p^{*N} \equiv \epsilon p^N$ , which is assumed to be of order  $\epsilon^0 = 1$ . We then find that we can transform the phase space of the Liouvillian of the system and replace  $p^N/m$  by  $\epsilon(p^{*N}/\mu)$  and  $\nabla_{p^N}$  by  $\epsilon \nabla_{p^{*N}}$ , to obtain

$$L = \epsilon(L_i^* + \nabla_{r^N} \phi \cdot \nabla_{p^{*N}}) + L_i + \nabla_{\xi^N} \phi \cdot \nabla_{\pi^N},$$

where  $L_i^* = -(\mathbf{p}^{*N}/\mu) \cdot \nabla_{r^N} + \nabla_{r^N} U \cdot \nabla_{p^{*N}}$ . For times longer than the isolated internal relaxation time,  $\tau_B$ , and to second order in powers of  $\epsilon$ , we obtain [7,8]

$$\begin{aligned} \dot{W}(X_i, t) = & \left[ \epsilon \left( -\frac{p^{*N}}{\mu} \cdot \nabla_{r^N} + \nabla_{r^N}(U + \omega) \cdot \nabla_{p^{*N}} \right) \right. \\ & \left. + \epsilon^2 \Gamma(r^N) : \nabla_{p^{*N}} \left( \nabla_{p^{*N}} + \frac{\beta p^{*N}}{\mu} \right) \right] W(t) + \mathcal{O}(\epsilon^3) \equiv OW(t), \end{aligned} \quad (2.24)$$

where the last equality is valid to order  $\epsilon^3$ . In eq. (2.24),  $\Gamma(\mathbf{r}^N)$  represents

$$\Gamma(\mathbf{r}^N) \equiv \int_0^\infty d\tau \langle \widehat{\nabla}_{\mathbf{r}^N} \phi \{ \exp[(L_i + \nabla_{\xi^N} \phi \cdot \nabla_{\pi^N})\tau] \widehat{\nabla}_{\mathbf{r}^N} \phi \} \rangle_i. \quad (2.25)$$

Since the potential  $\phi$  is short-ranged and  $\nabla_{r_i} \phi(r_{jk}) = \nabla_{r_{jk}} \phi(r_{jk})$ , we may write

$$\begin{aligned} & \langle \widehat{\nabla}_{r_{jk}} \phi \{ \exp[(L_i + \nabla_{\xi^N} \phi \cdot \nabla_{\pi^N})\tau] \widehat{\nabla}_{r_{lm}} \phi \} \rangle_i \\ & \approx \langle \widehat{\nabla}_{r_{jk}} \phi \{ \exp[(L_i + \nabla_{\xi^N} \phi \cdot \nabla_{\pi^N})\tau] \widehat{\nabla}_{r_{jk}} \phi \} \rangle_i \\ & \equiv \gamma_{jk}(r_{jk}, \tau) \hat{r}_{jk} \hat{r}_{jk}, \end{aligned} \quad (2.26)$$

and hence

$$\begin{aligned} & \epsilon^2 \Gamma(\mathbf{r}^N) : \nabla_{\mathbf{p}^{*N}} (\nabla_{\mathbf{p}^{*N}} + \beta \mathbf{p}^{*N} / \mu) \\ & \approx \frac{\epsilon^2}{2} \sum_j \sum_{k \neq j} \gamma_{jk} \hat{r}_{jk} \hat{r}_{jk} : (\nabla_{\mathbf{p}_j^*} - \nabla_{\mathbf{p}_k^*}) \left( (\nabla_{\mathbf{p}_j^*} - \nabla_{\mathbf{p}_k^*}) + \frac{\beta(\mathbf{p}_j^* - \mathbf{p}_k^*)}{\mu} \right), \end{aligned}$$

where  $\gamma_{jk} \equiv \int_0^\infty d\tau \gamma_{jk}(r_{jk}, \tau)$ . Eq. (2.24) defines the effective Liouvillian operator  $O$  for the distribution function  $W(t)$ . It can be shown that any arbitrary function  $G(X_i^*, t)$  of the translational phase space  $X_i^* \equiv (\mathbf{r}^N, \mathbf{p}^{*N})$  obeys the generalized Langevin equation to order  $\epsilon^3$ ,

$$\dot{G}(X_i^*, t) = K_G(t) + O^\dagger G(X_i^*, t), \quad (2.27)$$

where

$$O^\dagger = \epsilon \left( \frac{\mathbf{p}^{*N}}{m} \cdot \nabla_{\mathbf{r}^N} - \nabla_{\mathbf{r}^N} (U + \omega) \cdot \nabla_{\mathbf{p}^{*N}} \right) - \epsilon^2 \Gamma(\mathbf{r}^N) : \left( \frac{\beta}{\mu} \mathbf{p}^{*N} \nabla_{\mathbf{p}^{*N}} - \nabla_{\mathbf{p}^{*N}} \nabla_{\mathbf{p}^{*N}} \right) \quad (2.28)$$

and

$$K_G(t) \equiv -e^{-(1-\tilde{\Phi})Lt} (1 - \tilde{\Phi}) LG,$$

with  $\tilde{\Phi}B \equiv \int dX_i \tilde{\rho}B$ . Note that due to the terms proportional to  $\Gamma(\mathbf{r}^N)$  in  $O$  and  $O^\dagger$ ,  $O \neq -O^\dagger$ . It is important to note that the orders of magnitude of the various terms in  $O$  and  $O^\dagger$  are not really properly described by the factors of  $\epsilon$  multiplying them. There are other parameters which must be taken into

account such as the radii of the particles and the strengths of the interactions,  $U$  and  $\phi$ .

In the instance in which there is a separation in timescales between the relaxation of the momentum and spatial degrees of freedom, a relatively simple equation for the time dependence of the distribution for the more slowly relaxing variable can be obtained. When the momentum relaxation of the granular particles is faster than the relaxation in coordinate space, the Smoluchowski equation, which determines the time dependence of the coordinate space distribution function  $R(\mathbf{r}^N, t)$ , applies for times longer than the momentum relaxation time  $\tau_p$ , which is proportional to the mean free time of the granular particles. In general, moments of the distribution function  $W(\mathbf{r}^N, \mathbf{p}^N, t)$  rapidly approach those of the local equilibrium distribution function  $\sigma(\mathbf{r}^N, \mathbf{p}^N, t)$ , which corresponds to a system with locally defined chemical potential and temperature. The local equilibrium averages then relax on a much longer timescale to their equilibrium values in which the chemical potential and temperature are uniform. When there is friction in the system, however, the momentum distribution approaches a Maxwellian more rapidly since the frictional forces between particles tend to reduce the magnitude of their relative momenta. We shall represent the situation in which the spatial distribution remains relatively uniform over the timescale of the momentum relaxation by assigning a small parameter  $\delta$  to spatial derivatives of the reduced distribution  $R(t)$ .

The equilibrium coordinate space distribution is given by

$$R_e(\mathbf{r}^N) = \int d\mathbf{p}^N W_e(\mathbf{r}^N, \mathbf{p}^N). \quad (2.29)$$

The equilibrium conditional distribution function for the momenta is given by

$$\tilde{\Pi}_e(\mathbf{p}^N) \equiv \frac{W_e(\mathbf{r}^N, \mathbf{p}^N)}{R_e(\mathbf{r}^N)} = \frac{\exp[-\beta(\mathbf{p}^N \cdot \mathbf{p}^N)/2m]}{\int d\mathbf{p}^N \exp[-\beta(\mathbf{p}^N \cdot \mathbf{p}^N)/2m]}. \quad (2.30)$$

Following the development of the Smoluchowski equation of Oppenheim and McBride [9], we define the projection operator  $\hat{\mathcal{P}}$  by

$$\hat{\mathcal{P}}B = \tilde{\Pi}_e \int d\mathbf{p}^N B. \quad (2.31)$$

Note that

$$\hat{\mathcal{P}}W(\mathbf{r}^N, \mathbf{p}^N, t) = \tilde{\Pi}_e R(\mathbf{r}^N, t),$$

so that  $\int d\mathbf{p}^N \hat{\phi} W(t) = R(t)$ . Following the same procedures we used to obtain  $W(t)$  from  $\rho(t)$ , we obtain the exact equation [9]

$$\begin{aligned} \dot{R}(t) = & \frac{\delta}{m} \nabla_{r^N} \cdot \int d\mathbf{p}^N \mathbf{p}^N e^{(1-\hat{\phi})O t} (1 - \hat{\phi}) W(0) \\ & + \frac{\delta^2}{m^2} \nabla_{r^N} \cdot \int_0^t d\tau \int d\mathbf{p}^N \mathbf{p}^N e^{(1-\hat{\phi})O \tau} \tilde{\Pi}_e \mathbf{p}^N \cdot (\nabla_{r^N} - \nabla_{r^N} \ln R_e) R(t - \tau). \end{aligned} \tag{2.32}$$

From eq. (2.24), we see that

$$O = \delta O_e + O_D, \tag{2.33}$$

where

$$\begin{aligned} O_e = & \frac{-\mathbf{p}^N}{m} \cdot \nabla_{r^N} + \nabla_{r^N} (U + \omega) \cdot \nabla_{\mathbf{p}^N}, \\ O_D = & \Gamma(\mathbf{r}^N) : \nabla_{\mathbf{p}^N} \left( \nabla_{\mathbf{p}^N} + \frac{\beta \mathbf{p}^N}{m} \right), \end{aligned} \tag{2.34}$$

and we have associated a factor of  $\delta$  with each spatial derivative. Assuming that  $\delta \ll \epsilon$ , and noting that  $e^{O_D t} (1 - \hat{\phi}) W(0) = 0$  for  $t > \tau_p$ , we can write eq. (2.32) to order  $\delta^2$  and for  $t > \tau_p$  as

$$\dot{R}(\mathbf{r}^N, t) = \nabla_{r^N} \cdot \int_0^\infty d\tau \left( \int d\mathbf{p}^N \frac{\mathbf{p}^N}{m} e^{O_D \tau} \tilde{\Pi}_e \frac{\mathbf{p}^N}{m} \right) \cdot [\nabla_{r^N} + \beta \nabla_{r^N} (U + \omega)] R(t). \tag{2.35}$$

Expanding the exponential  $e^{O_D t}$  and calculating term by term, we find that

$$\begin{aligned} e^{O_D t} (\tilde{\Pi}_e \mathbf{p}^N) = & \tilde{\Pi}_e \mathbf{p}^N \cdot \left( \mathbb{1} - \frac{\beta}{m} \Gamma(\mathbf{r}^N) \tau + \frac{\beta^2 \tau^2}{2m^2} \Gamma(\mathbf{r}^N) \cdot \Gamma(\mathbf{r}^N) + \dots \right) \\ = & \tilde{\Pi}_e \mathbf{p}^N \cdot e^{-(\beta/m) \Gamma(\mathbf{r}^N) \tau}, \end{aligned} \tag{2.36}$$

and therefore

$$\dot{R}(\mathbf{r}^N, t) = \frac{K_B T}{m} \int_0^\infty d\tau \nabla_{r^N} \cdot e^{-(\beta/m) \Gamma(\mathbf{r}^N) \tau} \cdot [\nabla_{r^N} + \beta \nabla_{r^N} (U + \omega)] R(t). \tag{2.37}$$

Since the Hamiltonian of the system is both translationally and rotationally invariant,  $\omega(\mathbf{r}^N)$  depends only on the scalar distance between particles. For example, the Smoluchowski equation for  $R^{(2)}(\mathbf{r}_1, \mathbf{r}_2, t) \equiv \int d\mathbf{r}_3 \cdots d\mathbf{r}_N R(\mathbf{r}^N, t)$  to lowest order in the density is

$$\begin{aligned} \dot{R}^{(2)}(\mathbf{r}^N, t) = & \frac{K_B T}{m} \int_0^\infty d\tau \nabla_{\mathbf{r}_{12}} \cdot [(\mathbb{1} - \hat{\mathbf{r}}_{12} \hat{\mathbf{r}}_{12}) + \hat{\mathbf{r}}_{12} \hat{\mathbf{r}}_{12} e^{-2(\beta/m)\gamma(r_{12})\tau}] \\ & \cdot [\nabla_{\mathbf{r}_{12}} + \beta \nabla_{\mathbf{r}_{12}} \alpha(r_{12})] R^{(2)}(\mathbf{r}_1, \mathbf{r}_{12}, t), \end{aligned} \quad (2.38)$$

where

$$\alpha(r_{12}) = U(r_{12}) + \omega(r_{12}). \quad (2.39)$$

If we define  $g(\mathbf{r}, t) \equiv \int d\mathbf{r}_1 d\hat{\mathbf{r}} R^{(2)}(\mathbf{r}_1, \mathbf{r}, t)$  where we have set  $\mathbf{r}_{12} = \mathbf{r}$ , then to lowest order in the density

$$\begin{aligned} \dot{g}(\mathbf{r}, t) = & \frac{K_B T}{m} \int_0^\infty d\tau \int d\hat{\mathbf{r}} \nabla_{\mathbf{r}} \cdot [(\mathbb{1} - \hat{\mathbf{r}}\hat{\mathbf{r}}) + \hat{\mathbf{r}}\hat{\mathbf{r}} e^{-2(\beta/m)\gamma(r)\tau}] \\ & \cdot [\nabla_{\mathbf{r}} + \beta \hat{\mathbf{r}}\alpha'(r)] g(\mathbf{r}, t), \end{aligned} \quad (2.40)$$

where  $f'(r) \equiv (d/dr)f(r)$ . Carrying out the integrals over the angular part of  $\mathbf{r}$  and over  $\tau$ , yields

$$\begin{aligned} \frac{\dot{g}(r, t)}{(K_B T)^2} = & \frac{2}{r} \frac{1}{2\gamma(r)} [g'(r, t) + \beta\alpha'(r) g(r, t)] \\ & + \frac{d}{dr} \left( \frac{1}{2\gamma(r)} [g'(r, t) + \beta\alpha'(r) g(r, t)] \right). \end{aligned} \quad (2.41)$$

If we set

$$z(r, t) = \frac{1}{2\gamma(r)} [g'(r, t) + \beta\alpha'(r) g(r, t)],$$

then

$$\dot{g}(r, t) = (2/r)z(r, t) + z'(r, t). \quad (2.42)$$

The stationary solution of (2.41) occurs when  $\dot{g}(r, t) = 0$ , so that

$$(2/r)z(r) + z'(r) = 0. \quad (2.43)$$

One solution of (2.43) is  $z(r) = 0$ , which implies that  $g(r) = c e^{-\beta\alpha(r)}$ . However, there is another solution for nonzero  $z(r)$ , which is  $z(r) = K/r^2$ . If we model the potentials by

$$\alpha(r) = U(r) + \omega(r) = a(\sigma/r)^n, \tag{2.44}$$

where  $\sigma$  is the diameter of the particles,  $a$  is a constant, and  $n \gg 1$  so that

$$\alpha(r) \rightarrow \begin{cases} 0 & \text{if } r > \sigma, \\ \infty & \text{if } r < \sigma, \\ a & \text{if } r = \sigma, \end{cases} \tag{2.45}$$

then we find that  $g(r) = c\delta(r - \sigma)$ , which implies that

$$g(r) = \frac{\delta(r - \sigma)}{4\pi\sigma^2}. \tag{2.46}$$

Eq. (2.46) is, of course, not the equilibrium form for  $g(r)$ . Because the friction coefficient is essentially infinite on contact, the diffusion constant becomes zero and the system gets caught in a non-equilibrium part of phase space.

### 3. The nonlinear hydrodynamic equations

In this section, we derive the nonlinear hydrodynamic equations for the granular flow system based on projection operator techniques developed by Levine and Oppenheim [5]. Following their development, we define a special set of variables  $A(\mathbf{r}, X_t(t)) \equiv A(\mathbf{r}, t)$  to be the number density  $N(\mathbf{r}, t)$ , energy density  $E(\mathbf{r}, t)$  and momentum density  $\mathbf{P}(\mathbf{r}, t)$  and define a column vector  $\mathbf{C}(\mathbf{r}, t)$  to be

$$\mathbf{C}(\mathbf{r}, t) = \begin{pmatrix} 1 \\ \hat{A}(\mathbf{r}, t) \end{pmatrix}, \tag{3.1}$$

where  $\hat{A}(\mathbf{r}, t) \equiv A(\mathbf{r}, t) - \langle A(\mathbf{r}) \rangle$ . Here, the angle brackets  $\langle G(\mathbf{r}) \rangle$  denote the equilibrium average of an arbitrary function  $G(\mathbf{r})$  of the phase point  $X_t$ ,

$$\langle G(\mathbf{r}) \rangle \equiv \int dX_t W_e G(\mathbf{r}). \tag{3.2}$$

The non-equilibrium average of an arbitrary function  $G$  of  $X_t$  is defined by

$$g(t) \equiv \overline{G(t)} \equiv \int dX_t G(X_t) W(t), \tag{3.3}$$

where  $W(t)$  is the reduced distribution for the translational degrees of freedom (see eq. (2.10)). We now define the local equilibrium distribution function  $\sigma(t)$  to be

$$\sigma(t) \equiv \frac{\exp[\phi(\mathbf{r}, t) * C(\mathbf{r})] W_e(X_t)}{\int dX_t W_e(X_t) \exp[\phi(\mathbf{r}, t) * C(\mathbf{r})]}, \quad (3.4)$$

where the  $*$  product notation implies an integration over the repeated spatial argument  $\mathbf{r}$  as well as a vector product,

$$\phi(\mathbf{r}, t) * C(\mathbf{r}) \equiv \sum_n \int d\mathbf{r} \phi_n(\mathbf{r}, t) C_n(\mathbf{r}). \quad (3.5)$$

The  $\phi_n(\mathbf{r}, t)$  are chosen such that

$$\overline{C(\mathbf{r}, t)} \equiv \int dX_t C(\mathbf{r}) W(t) = \int dX_t C(\mathbf{r}) \sigma(t) \equiv \langle C(\mathbf{r}) \rangle_t, \quad (3.6)$$

where  $\langle \dots \rangle_t$  denotes the average over the local equilibrium distribution function. It follows from eq. (3.6) that the exact values of the non-equilibrium averages of the special variables in  $C(\mathbf{r})$  at any time  $t$  can be obtained from averaging these variables over the local equilibrium distribution function  $\sigma(t)$ . The  $\phi_n(\mathbf{r}, t)$  can be considered as Lagrange multipliers which fix the average of the densities in  $C(\mathbf{r})$  at each point in space  $\mathbf{r}$ , at time  $t$ . Note that the  $\phi$  conjugate to  $C = 1$  is zero.

We now define the projection operator  $\mathcal{P}(t)$ , which acts on an arbitrary dynamical variable  $G(\mathbf{r})$ , by

$$\mathcal{P}(t) G(\mathbf{r}) \equiv \langle G(\mathbf{r}) C(\mathbf{r}_1) \rangle_t * K_t^{-1}(\mathbf{r}_1, \mathbf{r}_2) * C(\mathbf{r}_2), \quad (3.7)$$

where

$$K_t(\mathbf{r}_1, \mathbf{r}_2) \equiv \langle C(\mathbf{r}_1) C(\mathbf{r}_2) \rangle_t.$$

Note that  $\overline{\mathcal{P}(t) G(\mathbf{r})} = \langle G(\mathbf{r}) \rangle_t$  since  $\langle C(\mathbf{r}) \rangle_t = \langle C(\mathbf{r}) 1 \rangle_t$ . We also define the projection operator  $\mathcal{P}^\dagger(t)$  by

$$\mathcal{P}^\dagger(t) G(\mathbf{r}) \equiv \left[ \int dX_t G(\mathbf{r}) C(\mathbf{r}_1) \right] * K_t^{-1}(\mathbf{r}_1, \mathbf{r}_2) * C(\mathbf{r}_2) \sigma(t).$$

$\mathcal{P}^\dagger(t)$  projects the distribution  $W(t)$  onto the local equilibrium distribution function  $\sigma(t)$ ,

$$\mathcal{P}^\dagger(t) W(t) = \sigma(t).$$

Furthermore, as was shown in Levine and Oppenheim [5],  $\mathcal{P}^\dagger(t)$  is the Hermitean adjoint of  $\mathcal{P}(t)$ , so that for arbitrary functions  $D(\mathbf{r}_1)$  and  $G(\mathbf{r})$ ,

$$\int dX_t D(\mathbf{r}_1) [\mathcal{P}(t) G(\mathbf{r})] = \int dX_t [\mathcal{P}^\dagger(t) D(\mathbf{r}_1)] G(\mathbf{r}). \tag{3.8}$$

We are interested in obtaining nonlinear equations of motion for the nonequilibrium averages  $a(\mathbf{r}, t)$  of the densities of the components in the set  $A(\mathbf{r})$ . Since  $a(\mathbf{r}, t) = \int dX_t A(\mathbf{r}) \sigma(t)$  by eq. (3.6), we obtain

$$\dot{a}(\mathbf{r}, t) = \int dX_t A(\mathbf{r}) \dot{\sigma}(t). \tag{3.9}$$

From the definition of  $\sigma(t)$ , it immediately follows that

$$\dot{\sigma}(t) = \tilde{A} * \dot{\phi}(t) \sigma(t), \tag{3.10}$$

where  $\tilde{A} \equiv A - \langle A \rangle_t$ , and hence

$$\begin{aligned} \mathcal{P}^\dagger(t) \dot{W}(t) &= \overline{C}^\dagger(t) * K_t^{-1} * C \sigma(t) = \dot{\phi}(t) * \langle \tilde{A} C \rangle_t * K_t^{-1} * C \sigma(t) \\ &= \dot{\phi}(t) * \tilde{A} \sigma(t) = \dot{\sigma}(t). \end{aligned} \tag{3.11}$$

If we define  $\chi(t)$  to be

$$\chi(t) \equiv [1 - \mathcal{P}^\dagger(t)] W(t) \equiv \mathcal{Q}^\dagger(t) W(t) = W(t) - \sigma(t), \tag{3.12}$$

so that  $W(t) = \mathcal{P}^\dagger(t) W(t) + \mathcal{Q}^\dagger(t) W(t) = \sigma(t) + \chi(t)$ , then it follows from section 2 and eq. (3.11) that

$$\dot{\sigma}(t) = \mathcal{P}^\dagger(t) O W(t) = \mathcal{P}^\dagger(t) O[\sigma(t) + \chi(t)]. \tag{3.13}$$

Now  $\dot{W}(t) = O W(t) = \mathcal{P}^\dagger(t) O W(t) + \mathcal{Q}^\dagger(t) O W(t)$ , so

$$\dot{\chi}(t) = \dot{W}(t) - \dot{\sigma}(t) = \mathcal{Q}^\dagger(t) O W(t) = \mathcal{Q}^\dagger(t) O \sigma(t) + \mathcal{Q}^\dagger(t) O \chi(t). \tag{3.14}$$

The formal solution of eq. (3.14) is

$$\begin{aligned} \chi(t) &= \left[ T_+ \exp \left( \int_0^t d\sigma \mathcal{Q}^\dagger(\sigma) O \right) \right] \chi(0) \\ &\quad + \int_0^t d\tau \left[ T_+ \exp \left( \int_\tau^t d\sigma \mathcal{Q}^\dagger(\sigma) O \right) \right] \mathcal{Q}^\dagger(\tau) O \sigma(\tau), \end{aligned} \tag{3.15}$$

where  $T_+$  is the time ordering operator. If we insert (3.15) in eq. (3.13), we obtain

$$\begin{aligned} \dot{\sigma}(t) = & \mathcal{P}^\dagger(t) O\sigma(t) + \mathcal{P}^\dagger(t) O \int_0^t d\tau \left[ T_+ \exp\left(\int_\tau^t d\sigma \mathcal{Q}^\dagger(\sigma) O\right) \right] \mathcal{Q}^\dagger(\tau) O\sigma(\tau) \\ & + \mathcal{P}^\dagger(t) O \left[ T_+ \exp\left(\int_0^t d\sigma \mathcal{Q}^\dagger(\sigma) O\right) \right] \chi(0), \end{aligned} \quad (3.16)$$

and hence

$$\begin{aligned} \dot{a}(\mathbf{r}, t) = & \int dX_t A(\mathbf{r}) \mathcal{P}^\dagger(t) O\sigma(t) + \int dX_t A(\mathbf{r}) \mathcal{P}^\dagger(t) O \int_0^t d\tau \\ & \times \left[ T_+ \exp\left(\int_\tau^t d\sigma \mathcal{Q}^\dagger(\sigma) O\right) \right] \mathcal{Q}^\dagger(\tau) O\sigma(\tau) + \chi(0) \text{ term}. \end{aligned} \quad (3.17)$$

For simplicity, we will assume that the initial distribution  $W(0)$  is of the local equilibrium form  $W(0) = \sigma(0)$  so that  $\chi(0) = 0$ . In general, this is not the case but we expect that the correlations due to the initial state of the system persist only for a microscopic time  $\tau_m$ , the time between the collisions of the granular particles. Thus, for time scales longer than this collision time,  $t \gg \tau_m$ , we can ignore the additional terms due to  $\chi(0)$ .

Now since  $\int dX_t A \mathcal{P}^\dagger(t) B(\mathbf{r}) = \int dX_t B(\mathbf{r}) A$ , for times longer than  $\tau_m$  we may rewrite eq. (3.17) as

$$\begin{aligned} \dot{a}(\mathbf{r}, t) = & \int dX_t A [O\sigma(t)] \\ & + \int dX_t A(\mathbf{r}) O \int_0^t d\tau \left[ T_+ \exp\left(\int_\tau^t d\sigma \mathcal{Q}^\dagger(\sigma) O\right) \right] \mathcal{Q}^\dagger(\tau) O\sigma(\tau), \\ = & \int dX_t (O^\dagger A) \sigma(t) \\ & + \int dX_t (O^\dagger A) \int_0^t d\tau \left[ T_+ \exp\left(\int_\tau^t d\sigma \mathcal{Q}^\dagger(\sigma) O\right) \right] \mathcal{Q}^\dagger(\tau) O\sigma(\tau), \end{aligned} \quad (3.18)$$

where  $O^\dagger$  is given by eq. (2.28).

We may reexpress  $O$  and  $O^\dagger$  in the form

$$\begin{aligned}
 O &= \left( \frac{-\mathbf{p}^N}{m} \cdot \nabla_{\mathbf{r}^N} + \nabla_{\mathbf{r}^N}(U + \omega) \cdot \nabla_{\mathbf{p}^N} \right) + \Omega_1 + \Omega_2, \\
 O^\dagger &= \left( \frac{\mathbf{p}^N}{m} \cdot \nabla_{\mathbf{r}^N} - \nabla_{\mathbf{r}^N}(U + \omega) \cdot \nabla_{\mathbf{p}^N} \right) + \Omega_1^\dagger + \Omega_2^\dagger \equiv O_1^\dagger + \Omega_2,
 \end{aligned}
 \tag{3.19}$$

where

$$\begin{aligned}
 \Omega_1 &= \frac{1}{2} \sum_j \sum_{k \neq j} \gamma_{jk} \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} : \left( \frac{\beta}{m} (\nabla_{\mathbf{p}_j} - \nabla_{\mathbf{p}_k})(\mathbf{p}_j - \mathbf{p}_k) \right), \\
 \Omega_2 &= \frac{1}{2} \sum_j \sum_{k \neq j} \gamma_{jk} \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} : (\nabla_{\mathbf{p}_j} - \nabla_{\mathbf{p}_k})(\nabla_{\mathbf{p}_j} - \nabla_{\mathbf{p}_k})
 \end{aligned}
 \tag{3.20}$$

and

$$\Omega_1^\dagger = \frac{-1}{2} \sum_j \sum_{k \neq j} \gamma_{jk} \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} : \left( \frac{\beta}{m} (\mathbf{p}_j - \mathbf{p}_k)(\nabla_{\mathbf{p}_j} - \nabla_{\mathbf{p}_k}) \right).
 \tag{3.21}$$

Note that  $(\Omega_1 + \Omega_1^\dagger)B = \sum_j \sum_{k \neq j} \gamma_{jk} (\beta/m)B$  and hence

$$\begin{aligned}
 O &= -L - \Omega_1^\dagger + (\Omega_1 + \Omega_1^\dagger) + \Omega_2 \\
 &= -O_1^\dagger + \sum_j \sum_{k \neq j} \left( \gamma_{jk} \frac{\beta}{m} \right) + \frac{1}{2} \sum_j \sum_{k \neq j} \gamma_{jk} \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} : (\nabla_{\mathbf{p}_j} - \nabla_{\mathbf{p}_k})(\nabla_{\mathbf{p}_j} - \nabla_{\mathbf{p}_k}).
 \end{aligned}
 \tag{3.22}$$

Inserting (3.22) into eq. (3.18), we obtain

$$\dot{a}(\mathbf{r}, t) = \langle O^\dagger A \rangle_t + \int_0^t d\tau \left\langle (O^\dagger A) \left[ T_+ \exp \left( \int_\tau^t d\sigma \mathcal{Q}^\dagger(\sigma) O \right) \right] \mathcal{Q}^\dagger(\tau) \psi(\tau) \right\rangle_\tau,
 \tag{3.23}$$

where

$$\begin{aligned}
 O\sigma(t) &= \psi(t) \sigma(t) \\
 &= \left[ (-O_1^\dagger A) * \phi(t) + \sum_j \sum_{k \neq j} \left( \frac{\beta}{m} \gamma_{jk} \right) \right. \\
 &\quad + \frac{1}{2} \sum_j \sum_{k \neq j} \gamma_{jk} \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} : [(\nabla_{\mathbf{p}_j} - \nabla_{\mathbf{p}_k})(\nabla_{\mathbf{p}_j} - \nabla_{\mathbf{p}_k}) A * \phi(t) \\
 &\quad \left. + (\nabla_{\mathbf{p}_j} - \nabla_{\mathbf{p}_k}) A * \phi(t) (\nabla_{\mathbf{p}_j} - \nabla_{\mathbf{p}_k}) A * \phi(t) \right] \sigma(t).
 \end{aligned}
 \tag{3.24}$$

Furthermore, since

$$\mathcal{Q}^\dagger(\tau) O\sigma(\tau) = [\psi(\tau) - \langle \psi(\tau) C \rangle_\tau * K_\tau^{-1} * C] \sigma(\tau) = [\mathcal{Q}(\tau) \psi(\tau)] \sigma(\tau),$$

where  $\mathcal{Q}(\tau) D \equiv D - \langle DC \rangle_\tau * K_\tau^{-1} * C$ , and

$$\int dX_t D \mathcal{Q}^\dagger(\tau) O[\psi(\tau) \sigma(\tau)] = \int dX_t [O^\dagger \mathcal{Q}(\tau) D] \psi(\tau) \sigma(\tau),$$

we may rewrite (3.23) as

$$\begin{aligned} \dot{a}(\mathbf{r}, t) &= \langle O^\dagger A \rangle_t \\ &+ \int_0^t d\tau \left\langle \left\{ \left[ T_- \exp\left( \int_\tau^t d\sigma \mathcal{Q}(\sigma) O^\dagger \right) \right] \mathcal{Q}(\tau) O^\dagger A \right\} [\mathcal{Q}(\tau) \psi(\tau)] \right\rangle_\tau \\ &\approx \langle O^\dagger A \rangle_t + \int_0^\infty d\tau \langle \{ \exp[\mathcal{Q}(t) O^\dagger \tau] \mathcal{Q}(t) O^\dagger A \} \mathcal{Q}(t) \psi(t) \rangle_t. \end{aligned} \quad (3.25)$$

To obtain (3.25), we have assumed that the set of conserved densities which compose the set  $A$  are slowly varying compared to the time scale of the relaxation of the other translational variables,  $\tau_m$ . In ordinary hydrodynamics, the time derivative of the hydrodynamic variables which compose  $A$  are proportional to the gradients of the system and are therefore small for relatively uniform systems. For granular flow systems, however, the energy is not conserved and hence the time derivative of the energy density has terms which are not proportional to a gradient and not necessarily small, even for uniform systems. Therefore, we shall restrict our attention to cases when the energy density is slowly varying over the timescale  $\tau_m$ . This assumption limits the magnitude of  $\gamma_{jk}$  in the system evolution operators  $\Omega_1$  and  $\Omega_2$ .

For simplicity, we will ignore long time tails due to mode coupling terms and expect the time dependent correlation function in (3.25) to decay quickly since the slowly varying behavior of  $O^\dagger A$  and  $\phi(t)$  due to the slowly varying densities in  $A$  are removed by the projection operator  $\mathcal{Q}(t)$ . We shall address the effect of mode coupling on the granular flow system in a later paper.

The local equilibrium average  $\langle \cdots \rangle_t$  can be approximated by an average over a homogeneous distribution function to lowest order in the gradients of the system by using the fact the thermodynamic quantities which appear in  $\phi(\mathbf{r}, t)$  vary slowly in space [1,10]. The term  $A * \phi(t)$  which appears in  $\sigma(t)$  can be written as

$$\begin{aligned}
 A * \phi(t) &= \int d\mathbf{r}' A(\mathbf{r}') \cdot \phi(\mathbf{r}', t) \\
 &= \int d\mathbf{r}' A(\mathbf{r}') \cdot \phi(\mathbf{r}) + \int d\mathbf{r}' A(\mathbf{r}') \cdot [\phi(\mathbf{r}') - \phi(\mathbf{r})]. \tag{3.26}
 \end{aligned}$$

By expanding the second term on the right of (3.26), we may write the local equilibrium average of an arbitrary dynamical variable  $B(\mathbf{r})$  as

$$\begin{aligned}
 \langle B(\mathbf{r}) \rangle_t &= \frac{\langle B(\mathbf{r}) e^{A * \phi(t)} \rangle}{\langle e^{A * \phi(t)} \rangle} \\
 &= \frac{\langle B(\mathbf{r}) e^{A \cdot \phi(\mathbf{r}, t)} \rangle}{\langle e^{A \cdot \phi(\mathbf{r}, t)} \rangle} + \int d\mathbf{r}' \frac{\langle B(\mathbf{r}) \tilde{A}(\mathbf{r}') e^{A \cdot \phi(\mathbf{r}, t)} \rangle}{\langle e^{A \cdot \phi(\mathbf{r}, t)} \rangle} \cdot [\phi(\mathbf{r}') - \phi(\mathbf{r})] \\
 &\quad + \dots, \tag{3.27}
 \end{aligned}$$

where  $A \equiv \int d\mathbf{r} A(\mathbf{r})$  and  $\tilde{A}(\mathbf{r}) \equiv A(\mathbf{r}) - \langle A(\mathbf{r}) \rangle_t$ . The correlation function in the second term of (3.27) is zero for  $|\mathbf{r} - \mathbf{r}'| > \xi$  so that the overall term is of order  $(\xi/L)^2 \sim (\xi k)^2 \ll 1$ , where  $L$  is the length scale on which  $\phi(\mathbf{r}')$  varies. Thus, to a very good approximation, we may write

$$\langle B(\mathbf{r}) \rangle_t \approx \frac{\langle B \rangle_H(\mathbf{r}, t)}{V}, \tag{3.28}$$

where the notation  $\langle B \rangle_H(\mathbf{r}, t)$  indicates the average of  $B = \int d\mathbf{r}' B(\mathbf{r}')$  over the homogeneous distribution function in which the thermodynamic forces  $\phi(\mathbf{r}', t)$  are evaluated at point  $\mathbf{r}$ .

We now turn our attention to the explicit evaluation of the terms which appear in the Euler part of equation (3.25), which is given by  $\langle O^\dagger A \rangle_t$ . The equations of motion are for the quantities  $a(\mathbf{r}, t) \approx (1/V) \langle A \rangle_H(\mathbf{r}, t)$ , whose components are the number density  $n(\mathbf{r}, t)$ , the energy density  $e(\mathbf{r}, t)$ , and the momentum density  $\mathbf{p}(\mathbf{r}, t)$ . It is convenient to introduce the reduced momentum

$$\mathbf{p}_j^+ \equiv \mathbf{p}_j - m\mathbf{v}(\mathbf{r}_j) \tag{3.29}$$

and

$$\mathbf{P}^+(\mathbf{r}) \equiv \sum_j \mathbf{p}_j^+ \delta(\mathbf{r} - \mathbf{r}_j). \tag{3.30}$$

We then find that  $A \cdot \phi(\mathbf{r}, t) = -E^+ \beta(\mathbf{r}, t) + \beta(\mathbf{r}, t) \mu(\mathbf{r}, t) N$ , where  $E^+$  is obtained from  $E$  by substituting  $\mathbf{p}_j$  by  $\mathbf{p}_j^+$ , and hence

$$\begin{aligned} \mathbf{P}(\mathbf{r}) &= \mathbf{P}^+(\mathbf{r}) + m\mathbf{v}(\mathbf{r}) N(\mathbf{r}), \\ E(\mathbf{r}) &= E^+(\mathbf{r}) + \mathbf{v}(\mathbf{r}) \cdot \mathbf{P}^+(\mathbf{r}) + \frac{1}{2}m\mathbf{v}^2(\mathbf{r}) N(\mathbf{r}). \end{aligned} \quad (3.31)$$

Equation (3.25) is the fundamental equation which describes the nonlinear hydrodynamics of granular flow systems. Tedious but straightforward algebra yields

$$\begin{aligned} O^\dagger N(\mathbf{r}) &= -\frac{1}{m} \nabla_r \cdot [\mathbf{P}^+(\mathbf{r}) + m\mathbf{v}(\mathbf{r}) N(\mathbf{r})], \\ O^\dagger \mathbf{P}(\mathbf{r}) &= -\nabla_r \cdot [\tilde{\tau}^+(\mathbf{r}) + \mathbf{v}(\mathbf{r}) \mathbf{P}^+(\mathbf{r}) + \mathbf{P}^+(\mathbf{r}) \mathbf{v}(\mathbf{r}) + N(\mathbf{r}) m\mathbf{v}(\mathbf{r}) \mathbf{v}(\mathbf{r})] \\ O^\dagger E(\mathbf{r}) &= -\nabla_r \cdot [\tilde{J}_E^+(\mathbf{r}) + \mathbf{v}(\mathbf{r}) \cdot \tilde{\tau}^+(\mathbf{r}) + \mathbf{v}(\mathbf{r}) E^+(\mathbf{r}) + \frac{1}{2}v^2(\mathbf{r}) \mathbf{P}^+(\mathbf{r}) \\ &\quad + \mathbf{v}(\mathbf{r}) \mathbf{v}(\mathbf{r}) \cdot \mathbf{P}^+(\mathbf{r}) + \frac{1}{2}m\mathbf{v}^2(\mathbf{r}) N(\mathbf{r}) \mathbf{v}(\mathbf{r})] \\ &\quad - \sum_j \sum_{k \neq j} \frac{\gamma_{jk}}{m} \left( \frac{\beta}{2m} \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} : (\mathbf{p}_j^+ - \mathbf{p}_k^+) (\mathbf{p}_j^+ - \mathbf{p}_k^+) - 1 \right) \delta(\mathbf{r} - \mathbf{r}_j) \\ &\quad + 2\tau^{0+}(\mathbf{r}) : [\nabla_r \mathbf{v}(\mathbf{r})], \end{aligned} \quad (3.32)$$

where

$$\begin{aligned} \tau^+(\mathbf{r}) &= \sum_j \left( \frac{\mathbf{p}_j^+ \mathbf{p}_j^+}{m} + \frac{1}{2} \sum_{k \neq j} \mathbf{r}_{jk} \mathbf{F}_{jk} \right) \delta(\mathbf{r} - \mathbf{r}_j), \\ \tau^{0+}(\mathbf{r}) &= -\frac{\beta}{2m} \sum_j \sum_{k \neq j} (\gamma_{jk} r_{jk}) \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} \cdot (\mathbf{p}_j^+ - \mathbf{p}_k^+) \delta(\mathbf{r} - \mathbf{r}_j), \\ \tilde{\tau}^+(\mathbf{r}) &= \tau^+(\mathbf{r}) + \tau^{0+}(\mathbf{r}), \end{aligned} \quad (3.33)$$

and

$$\begin{aligned} J_E^+(\mathbf{r}) &= \sum_j \left( \frac{e_j \mathbf{p}_j^+}{m} + \frac{1}{2m} \sum_{k \neq j} \mathbf{r}_{jk} \mathbf{p}_j^+ \cdot \mathbf{F}_{jk} \right) \delta(\mathbf{r} - \mathbf{r}_j), \\ J_E^{0+}(\mathbf{r}) &= -\frac{\beta}{2m^2} \sum_j \sum_{k \neq j} (\gamma_{jk} r_{jk}) \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} : (\mathbf{p}_j^+ - \mathbf{p}_k^+) (\mathbf{p}_j^+ - \mathbf{p}_k^+) \delta(\mathbf{r} - \mathbf{r}_j), \\ \tilde{J}_E^+(\mathbf{r}) &= J_E^+(\mathbf{r}) + J_E^{0+}(\mathbf{r}). \end{aligned}$$

Thus, the Euler part  $\langle O^\dagger A \rangle_t$  yields

$$\begin{aligned} \dot{n}^E(\mathbf{r}, t) &= -\nabla_r \cdot [n(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t)], \\ \dot{\mathbf{p}}^E(\mathbf{r}, t) &= -\nabla_r P_h(\mathbf{r}, t) - \nabla_r \cdot [n(\mathbf{r}, t) m\mathbf{v}(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t)], \\ \dot{e}^E(\mathbf{r}, t) &= -\nabla_r \cdot [\mathbf{v}(\mathbf{r}, t) h^+(\mathbf{r}, t)] - \nabla_r \cdot [\frac{1}{2}m\mathbf{v}(\mathbf{r}, t) v^2(\mathbf{r}, t) n(\mathbf{r}, t)] \\ &\quad - \frac{\langle \sum_j \sum_{k \neq j} \gamma_{jk} \rangle_H(\mathbf{r}, t)}{mVT} [T(\mathbf{r}, t) - T], \end{aligned} \quad (3.34)$$

where we have used the facts that

$$\frac{\langle \tilde{\tau}^+ \rangle_{\text{H}}(\mathbf{r}, t)}{V} = P_{\text{h}}(\mathbf{r}, t) \mathbb{1},$$

where  $P_{\text{h}}(\mathbf{r}, t)$  is the hydrostatic pressure,  $\langle \mathbf{J}_{\text{E}}^+ \rangle_{\text{H}}(\mathbf{r}, t) = 0$ , and  $h^+(\mathbf{r}, t) = e^+(\mathbf{r}, t) + P_{\text{h}}(\mathbf{r}, t)$  is the internal enthalpy density. Note that unlike ordinary hydrodynamics in which  $\dot{e}(\mathbf{r}, t)$  is proportional to a gradient,  $\dot{e}(\mathbf{r}, t)$  has a term which exists for uniform systems, and reflects the fact that energy flows out of the system.

We now turn our attention to evaluating the second term in eq. (3.25), namely the term

$$\dot{a}^{\text{D}}(\mathbf{r}, t) \equiv \int_0^{\infty} d\tau \frac{1}{V} \langle [G(\tau) \mathcal{Q}(t) O^{\dagger} A] \mathcal{Q}(t) \psi(t) \rangle_{\text{H}}(\mathbf{r}, t), \quad (3.35)$$

where

$$G(\tau) \equiv e^{\mathcal{Q}(t) O^{\dagger} \tau}.$$

From eqs. (3.7) and (3.32), we see that

$$\mathcal{Q}(t) O^{\dagger} N(\mathbf{r}) = 0,$$

$$\mathcal{Q}(t) O^{\dagger} \mathbf{P}(\mathbf{r}) = -\mathcal{Q}(t) \nabla_{\mathbf{r}} \cdot \tilde{\tau}^+(\mathbf{r}),$$

$$\begin{aligned} \mathcal{Q}(t) O^{\dagger} E(\mathbf{r}) &= \mathcal{Q}(t) \left[ -\nabla_{\mathbf{r}} \cdot \tilde{\mathbf{J}}_{\text{E}}^+(\mathbf{r}) - \nabla_{\mathbf{r}} \cdot [\tilde{\tau}^+(\mathbf{r}) \cdot \mathbf{v}(\mathbf{r})] + 2\tau^{0+}(\mathbf{r}) : [\nabla_{\mathbf{r}} \mathbf{v}(\mathbf{r})] \right. \\ &\quad \left. - \sum_j \sum_{k \neq j} \frac{1}{m} \gamma_{jk} \left( \frac{\beta}{2m} \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} : (\mathbf{p}_j^+ - \mathbf{p}_k^+) (\mathbf{p}_j^+ - \mathbf{p}_k^+) - 1 \right) \right. \\ &\quad \left. \times \delta(\mathbf{r} - \mathbf{r}_j) \right] \\ &\equiv \mathcal{Q}(t) \{ -\nabla_{\mathbf{r}} \cdot \tilde{\mathbf{J}}_{\text{E}}^+(\mathbf{r}) - \nabla_{\mathbf{r}} \cdot [\tilde{\tau}^+(\mathbf{r}) \cdot \mathbf{v}(\mathbf{r})] \\ &\quad + 2\tau^{0+}(\mathbf{r}) : [\nabla_{\mathbf{r}} \mathbf{v}(\mathbf{r})] - X(\mathbf{r}) \}, \end{aligned}$$

and hence

$$\begin{aligned} \dot{n}^{\text{D}}(\mathbf{r}, t) &= 0, \\ \dot{e}^{\text{D}}(\mathbf{r}, t) &= -\nabla_{\mathbf{r}} \cdot \int_0^{\infty} d\tau \frac{1}{V} \langle [G(\tau) \mathcal{Q}(t) (\tilde{\mathbf{J}}_{\text{E}}^+ + \mathbf{v}(\mathbf{r}) \cdot \tilde{\tau}^+)] \mathcal{Q}(t) \psi(t) \rangle_{\text{H}}(\mathbf{r}, t) \\ &\quad + \int_0^{\infty} d\tau \frac{1}{V} \langle [G(\tau) \mathcal{Q}(t) X] \mathcal{Q}(t) \psi(t) \rangle_{\text{H}}(\mathbf{r}, t) \\ &\quad + 2\nabla_{\mathbf{r}} \mathbf{v}(\mathbf{r}) : \int_0^{\infty} d\tau \frac{1}{V} \langle [G(\tau) \mathcal{Q}(t) \tau^{0+}] \mathcal{Q}(t) \psi(t) \rangle_{\text{H}}(\mathbf{r}, t), \end{aligned}$$

$$\dot{\mathbf{p}}^D(\mathbf{r}, t) = -\nabla_r \cdot \int_0^\infty d\tau \frac{1}{V} \langle [G(\tau) \mathcal{Q}(t) \tilde{\tau}^+] \mathcal{Q}(t) \psi(t) \rangle_H(\mathbf{r}, t). \quad (3.36)$$

According to eq. (3.24),  $\psi(t)$  is given by

$$\begin{aligned} \psi(t) = & -(O_1^\dagger A) * \phi(\mathbf{r}, t) + \frac{1}{m} \sum_j \sum_{k \neq j} \gamma_{jk} [\beta - \beta(\mathbf{r}, t)] \\ & + \sum_j \sum_{k \neq j} \gamma_{jk} [\hat{\mathbf{r}}_{jk} \cdot (\nabla_{\mathbf{p}_j} - \nabla_{\mathbf{p}_k}) A * \phi(\mathbf{r}, t)]^2, \end{aligned}$$

and hence to linear order in the gradients of the system we obtain for  $\mathcal{Q}(t) \psi(t)$ :

$$\begin{aligned} \mathcal{Q}(t) \psi(t) = & \mathcal{Q}(t) \left[ (-\tau^+ + \tau^{0+}) : [\nabla_r \mathbf{v}(\mathbf{r})] \beta(\mathbf{r}, t) \right. \\ & + \mathbf{J}_E^+ \cdot \nabla_r \beta(\mathbf{r}, t) + \mathbf{J}_E^{0+} \cdot \nabla_r \beta(\mathbf{r}, t) \left( 1 - \frac{2\beta(\mathbf{r}, t)}{\beta} \right) \\ & \left. - Y[\beta - \beta(\mathbf{r}, t)] \right], \end{aligned} \quad (3.37)$$

where

$$Y \equiv \sum_j \sum_{k \neq j} \frac{\gamma_{jk}}{m} \left( \frac{\beta(\mathbf{r}, t)}{2m} \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} : (\mathbf{p}_j^+ - \mathbf{p}_k^+) (\mathbf{p}_j^+ - \mathbf{p}_k^+) - 1 \right).$$

Combining eqs. (3.36) and (3.37), we obtain

$$\begin{aligned} \dot{e}^D(\mathbf{r}, t) = & \int_0^\infty d\tau \frac{1}{V} \left\{ -\nabla_r \cdot \left\langle [G(\tau) \mathcal{Q}(t) \mathbf{J}_E^+] \mathcal{Q}(t) \left[ \mathbf{J}_E^+ + \mathbf{J}_E^{0+} \left( 1 - \frac{2\beta(\mathbf{r}, t)}{\beta} \right) \right] \right\rangle_H(\mathbf{r}, t) \right. \\ & \cdot \nabla_r \beta(\mathbf{r}, t) \\ & + \nabla_r \cdot \langle [G(\tau) \mathcal{Q}(t) [\tau^+ \cdot \mathbf{v}(\mathbf{r})]] \mathcal{Q}(t) (\tau^+ - \tau^{0+}) \rangle_H(\mathbf{r}, t) \\ & : [\nabla_r \mathbf{v}(\mathbf{r}, t)] \beta(\mathbf{r}, t) \\ & + \nabla_r \cdot \langle [G(\tau) \mathcal{Q}(t) [\tau^+ \cdot \mathbf{v}(\mathbf{r})]] \mathcal{Q}(t) Y \rangle_H(\mathbf{r}, t) [\beta - \beta(\mathbf{r}, t)] \\ & + 2[\nabla_r \mathbf{v}(\mathbf{r}, t)] : \langle [G(\tau) \mathcal{Q}(t) \tau^{0+}] \mathcal{Q}(t) (\tau^{0+} - \tau^+) \rangle_H(\mathbf{r}, t) \\ & : [\nabla_r \mathbf{v}(\mathbf{r}, t)] \beta(\mathbf{r}, t) \\ & \left. - 2[\nabla_r \mathbf{v}(\mathbf{r}, t)] : \{ [G(\tau) \mathcal{Q}(t) \tau^{0+}] \mathcal{Q}(t) Y \}_H(\mathbf{r}, t) [\beta - \beta(\mathbf{r}, t)] \right\} \end{aligned}$$

$$\begin{aligned}
 & - \langle [G(\tau) \mathcal{Q}(t) X] \mathcal{Q}(t) (\tau^{0+} - \tau^+) \rangle_{\text{H}}(\mathbf{r}, t) : [\nabla_r \mathbf{v}(\mathbf{r}, t)] \beta(\mathbf{r}, t) \\
 & + \langle [G(\tau) \mathcal{Q}(t) X] \mathcal{Q}(t) Y \rangle_{\text{H}}(\mathbf{r}, t) [\beta - \beta(\mathbf{r}, t)] \Big\}, \\
 \dot{p}^{\text{D}}(\mathbf{r}, t) = & \int_0^\infty d\tau \frac{1}{V} \{ \nabla_r \cdot \langle [G(\tau) \mathcal{Q}(t) (\tau^+ + \tau^{0+})] \mathcal{Q}(t) (\tau^+ - \tau^{0+}) \rangle_{\text{H}}(\mathbf{r}, t) \\
 & : \beta(\mathbf{r}, t) [\nabla_r \mathbf{v}(\mathbf{r}, t)] \\
 & + \nabla_r \cdot \langle [G(\tau) \mathcal{Q}(t) (\tau^+ + \tau^{0+})] \mathcal{Q}(t) Y \rangle_{\text{H}}(\mathbf{r}, t) [\beta - \beta(\mathbf{r}, t)] \Big\}. \tag{3.38}
 \end{aligned}$$

Due to the magnitude of the granular particle mass,  $m$ , each factor of  $\gamma_{jk}/m = \epsilon^2 \gamma_{jk}/\mu$  yields a small parameter  $\lambda$ . Therefore, we associate a factor of  $\lambda$  with each factor of  $\gamma_{jk}/m$  to characterize the order of magnitude of each term. If  $\lambda$  is on the order of the gradients of the system, then to second order in the gradients we may simplify equation (3.38) and obtain the nonlinear hydrodynamic equations

$$\begin{aligned}
 \dot{n}(\mathbf{r}, t) = & -\nabla_r \cdot [n(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t)], \\
 [e^+(\mathbf{r}, t) + \frac{1}{2} m v^2(\mathbf{r}, t) n(\mathbf{r}, t)] = & \\
 & -\nabla_r \cdot [\mathbf{v}(\mathbf{r}, t) h^+(\mathbf{r}, t)] - \nabla_r \cdot [\frac{1}{2} m \mathbf{v}(\mathbf{r}, t) v^2(\mathbf{r}, t) n(\mathbf{r}, t)] \\
 & + \int_0^\infty d\tau \frac{1}{V} \left( -\nabla_r \cdot \langle [G(\tau) \mathcal{Q}(t) J_{\text{E}}^+] \mathcal{Q}(t) J_{\text{E}}^+ \rangle_{\text{H}}(\mathbf{r}, t) \cdot \nabla_r \beta(\mathbf{r}, t) \right. \\
 & + \nabla_r \cdot \langle \{ G(\tau) \mathcal{Q}(t) [\tau^+ \cdot \mathbf{v}(\mathbf{r}, t)] \} \mathcal{Q}(t) \tau^+ \rangle_{\text{H}}(\mathbf{r}, t) : \beta(\mathbf{r}, t) [\nabla_r \mathbf{v}(\mathbf{r}, t)] \\
 & + \lambda \nabla_r \cdot \langle \{ G(\tau) \mathcal{Q}(t) [\tau^+ \cdot \mathbf{v}(\mathbf{r}, t)] \} \mathcal{Q}(t) Y \rangle_{\text{H}}(\mathbf{r}, t) [\beta - \beta(\mathbf{r}, t)] \\
 & + \lambda \langle [G(\tau) \mathcal{Q}(t) X] \mathcal{Q}(t) \tau^+ \rangle_{\text{H}}(\mathbf{r}, t) : [\nabla_r \mathbf{v}(\mathbf{r}, t)] \beta(\mathbf{r}, t) \\
 & + \lambda^2 \langle [G(\tau) \mathcal{Q}(t) X] \mathcal{Q}(t) Y \rangle_{\text{H}}(\mathbf{r}, t) [\beta - \beta(\mathbf{r}, t)] \Big) \\
 & - \frac{\lambda \langle \sum_j \sum_{k \neq j} \gamma_{jk} \rangle_{\text{H}}(\mathbf{r}, t)}{mV} \left( \frac{\beta}{\beta(\mathbf{r}, t)} - 1 \right),
 \end{aligned}$$

$$\begin{aligned}
 [m \mathbf{v}(\mathbf{r}, t) n(\mathbf{r}, t)] = & -\nabla_r P_{\text{h}}(\mathbf{r}, t) - \nabla_r \cdot [n(\mathbf{r}, t) m \mathbf{v}(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t)] \\
 & + \int_0^\infty d\tau \frac{1}{V} \{ \nabla_r \cdot \langle [G(\tau) \mathcal{Q}(t) \tau^+] \mathcal{Q}(t) \tau^+ \rangle_{\text{H}}(\mathbf{r}, t)
 \end{aligned}$$

$$\begin{aligned}
& :[\nabla_r \mathbf{v}(\mathbf{r}, t)]\beta(\mathbf{r}, t) \\
& + \lambda \nabla_r \cdot \langle [G(\tau) \mathcal{Q}(t) \tau^+] \mathcal{Q}(t) Y \rangle_{\text{H}}(\mathbf{r}, t) [\beta - \beta(\mathbf{r}, t)] \},
\end{aligned} \tag{3.39}$$

where

$$\begin{aligned}
X &= \sum_j \sum_{k \neq j} \frac{\gamma_{jk}}{m} \left( \frac{\beta}{2\mu} \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} : (\mathbf{p}_j^{+*} - \mathbf{p}_k^{+*})(\mathbf{p}_j^{+*} - \mathbf{p}_k^{+*}) - 1 \right), \\
Y &= \sum_j \sum_{k \neq j} \frac{\gamma_{jk}}{m} \left( \frac{\beta(\mathbf{r}, t)}{2\mu} \hat{\mathbf{r}}_{jk} \hat{\mathbf{r}}_{jk} : (\mathbf{p}_j^{+*} - \mathbf{p}_k^{+*})(\mathbf{p}_j^{+*} - \mathbf{p}_k^{+*}) - 1 \right).
\end{aligned} \tag{3.40}$$

Examining eq. (3.39), we see that the nonlinear hydrodynamic equations of motion for the granular flow system differ from those conservative systems by the additional four terms appearing at the end of the energy density equation and the additional last term in the momentum density equation. Note that the number density equation is unchanged.

Following the development of the ordinary hydrodynamic equations [5], we introduce the fourth rank tensor  $\mathbf{O}_p(\mathbf{r}, t)$  and the second rank tensor  $\mathbf{O}_e(\mathbf{r}, t)$  which are defined by

$$\begin{aligned}
[\mathbf{O}_p(\mathbf{r}, t)]_{ijkl} &= \int_0^\infty d\tau \frac{\langle [G(\tau) \mathcal{Q}(t) \tau_{ij}^+] \mathcal{Q}(t) \tau_{kl}^+] \rangle_{\text{H}}(\mathbf{r}, t)}{V} \\
&= \frac{1}{\beta(\mathbf{r}, t)} \{ [\zeta(\mathbf{r}, t) - \frac{2}{3}\eta(\mathbf{r}, t)] \delta_{ij} \delta_{kl} + \eta(\mathbf{r}, t) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \}, \\
[\mathbf{O}_e(\mathbf{r}, t)]_{ij} &= \int_0^\infty d\tau \frac{\langle [G(\tau) \mathcal{Q}(t) (\mathbf{J}_E^+)_i] \mathcal{Q}(t) (\mathbf{J}_E^+)_j \rangle_{\text{H}}(\mathbf{r}, t)}{V} \\
&= \frac{1}{k_B \beta^2(\mathbf{r}, t)} \kappa(\mathbf{r}, t) \delta_{ij}.
\end{aligned} \tag{3.41}$$

In eq. (3.41),  $k_B$  is Boltzmann's constant and  $\zeta(\mathbf{r}, t)$ ,  $\eta(\mathbf{r}, t)$  and  $\kappa(\mathbf{r}, t)$  are, respectively, the bulk viscosity, the shear viscosity and the thermal conductivity of a homogeneous system with number density  $n(\mathbf{r}, t)$  and internal energy density  $e^+(\mathbf{r}, t)$ . Inserting (3.41) into (3.39), we obtain the fundamental result of this section:

$$\begin{aligned}
 \dot{n}(\mathbf{r}, t) &= -\nabla_r \cdot [n(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t)], \\
 [e^+(\mathbf{r}, t) + \frac{1}{2} m \dot{v}^2(\mathbf{r}, t) n(\mathbf{r}, t)] &= \\
 &-\nabla_r \cdot \{ \mathbf{v}(\mathbf{r}, t) [h^+(\mathbf{r}, t) + \frac{1}{2} m n(\mathbf{r}, t) v^2(\mathbf{r}, t)] \} \\
 &+ \nabla_r \cdot \{ \mathbf{v}(\mathbf{r}, t) \cdot \mathbf{O}_p(\mathbf{r}, t) \beta(\mathbf{r}, t) : [\nabla_r \mathbf{v}(\mathbf{r}, t)] - \mathbf{O}_e(\mathbf{r}, t) \cdot \nabla_r \beta(\mathbf{r}, t) \} \\
 &+ \int_0^\infty d\tau \frac{1}{V} \left( \lambda \nabla_r \cdot \langle \{ \mathcal{G}(\tau) \mathcal{Q}(t) [\tau^+ \cdot \mathbf{v}(\mathbf{r})] \} \mathcal{Q}(t) Y \rangle_H(\mathbf{r}, t) [\beta - \beta(\mathbf{r}, t)] \right. \\
 &+ \lambda \langle [G(\tau) \mathcal{Q}(t) X] \mathcal{Q}(t) \tau^+ \rangle_H(\mathbf{r}, t) : [\nabla_r \mathbf{v}(\mathbf{r}, t)] \beta(\mathbf{r}, t) \\
 &+ \lambda^2 \langle [G(\tau) \mathcal{Q}(t) X] \mathcal{Q}(t) Y \rangle_H(\mathbf{r}, t) [\beta - \beta(\mathbf{r}, t)] \Big) \\
 &- \frac{\lambda \langle \Sigma_j \Sigma_{k \neq j} \gamma_{jk} \rangle_H(\mathbf{r}, t)}{mV} \left( \frac{\beta}{\beta(\mathbf{r}, t)} - 1 \right), \tag{3.42} \\
 [m \dot{v}(\mathbf{r}, t) n(\mathbf{r}, t)] &= -\nabla_r P_h(\mathbf{r}, t) - \nabla_r \cdot [n(\mathbf{r}, t) m \mathbf{v}(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t)] \\
 &+ \nabla_r \cdot \{ \mathbf{O}_p(\mathbf{r}, t) \beta(\mathbf{r}, t) : [\nabla_r \mathbf{v}(\mathbf{r}, t)] \} \\
 &+ \lambda \nabla_r \cdot \int_0^\infty d\tau \frac{1}{V} \langle [G(\tau) \mathcal{Q}(t) \tau^+] \mathcal{Q}(t) Y \rangle_H(\mathbf{r}, t) \\
 &\times [\beta - \beta(\mathbf{r}, t)].
 \end{aligned}$$

If  $\lambda^{1/2}$  is on the order of the gradients of the system, the momentum density equation for the granular flow system is identical to the ordinary momentum density equation to second order in the gradients of the system, and the energy density equation contains only the additional term

$$- \frac{\langle \Sigma_j \Sigma_{k \neq j} \gamma_{jk} \rangle_H(\mathbf{r}, t)}{mV} \left( \frac{\beta}{\beta(\mathbf{r}, t)} - 1 \right),$$

which accounts for the unidirectional flow of energy from the translational degrees of freedom in the granular flow system into the granular particle internal modes when  $T(\mathbf{r}, t) > T$ .

The effect of external stress on the clustering phenomenon discussed in section 2 is easily obtained from the hydrodynamic equations presented above.

If a steady state can be maintained which prevents the hydrodynamic temperature decaying to  $T$ , the internal mode temperature, the clustering phenomena will be vitiated. As a concrete example we consider the effect of an imposed shear on the system. There exists a steady state in which

$$\mathbf{v}(\mathbf{r}, t) = by\hat{\mathbf{e}}_x, \quad (3.43)$$

and  $P_h$  and  $n$  are independent of position. This implies that all the thermodynamic and transport coefficients are independent of  $\mathbf{r}$ . Under these conditions,  $\dot{n} = \dot{p} = 0$ , and the steady state form of the energy equation yields the condition

$$b^2\eta = \frac{(T_H - T)}{T} \left( \frac{\lambda\Lambda_1}{m} - \frac{\lambda^2\Lambda_2}{k_B T_H} \right). \quad (3.44)$$

(Refer to eq. (4.2) in section 4 for the definitions of  $\Lambda_1$  and  $\Lambda_2$ .) Here,  $T$  is again the temperature of the internal modes and  $T_H > T$  is the temperature that is maintained by the shear stress.  $\eta$  is the equilibrium value of the shear stress, defined in eq. (3.41).

#### 4. The linearized hydrodynamic equations

In this section we examine the system of hydrodynamic equations obtained by linearizing eq. (3.42), which may be written as

$$\begin{aligned} \dot{n}(\mathbf{r}, t) &= -\nabla_r \cdot [n(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t)] \\ [e^+(\mathbf{r}, t) + \frac{1}{2}mv^2(\mathbf{r}, t) n(\mathbf{r}, t)] &= \\ &-\nabla_r \cdot \{ \mathbf{v}(\mathbf{r}, t) [h^+(\mathbf{r}, t) + \frac{1}{2}mv^2(\mathbf{r}, t) n(\mathbf{r}, t)] \} + \nabla_r \cdot [ \kappa(\mathbf{r}, t) \nabla_r T(\mathbf{r}, t) ] \\ &+ \nabla_r \cdot \{ \mathbf{v}(\mathbf{r}, t) [ \zeta(\mathbf{r}, t) - \frac{2}{3}\eta(\mathbf{r}, t) ] \nabla_r \cdot \mathbf{v}(\mathbf{r}, t) \} \\ &+ \nabla_{r_i} [ v_j(\mathbf{r}, t) \eta(\mathbf{r}, t) \nabla_{r_i} v_j(\mathbf{r}, t) + v_j(\mathbf{r}, t) \eta(\mathbf{r}, t) \nabla_{r_j} v_i(\mathbf{r}, t) ] \\ &+ \lambda \nabla_r \cdot \{ \mathbf{v}(\mathbf{r}, t) \Theta_1(\mathbf{r}, t) [ \beta - \beta(\mathbf{r}, t) ] \} + \lambda [ \Theta_2(\mathbf{r}, t) \beta(\mathbf{r}, t) \nabla_r \cdot \mathbf{v}(\mathbf{r}, t) ] \\ &+ \lambda^2 \Lambda_2(\mathbf{r}, t) [ \beta - \beta(\mathbf{r}, t) ] - \lambda \frac{\Lambda_1(\mathbf{r}, t)}{m} \left( \frac{\beta}{\beta(\mathbf{r}, t)} - 1 \right), \end{aligned}$$

$$\begin{aligned}
 [m\mathbf{v}(\mathbf{r}, t) \dot{n}(\mathbf{r}, t)] &= -\nabla_r P_h(\mathbf{r}, t) - \nabla_r \cdot [n(\mathbf{r}, t) m\mathbf{v}(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t)] \\
 &+ \nabla_r \cdot \{[\zeta(\mathbf{r}, t) - \frac{2}{3}\eta(\mathbf{r}, t)]\nabla_r \cdot \mathbf{v}(\mathbf{r}, t)\} + \nabla_{r_i}[\eta(\mathbf{r}, t) \nabla_{r_i} \mathbf{v}(\mathbf{r}, t)] \\
 &+ \nabla_{r_i}[\eta(\mathbf{r}, t) \nabla_{r_i} v_i(\mathbf{r}, t)] + \lambda \nabla_r \Theta_1(\mathbf{r}, t) [\beta - \beta(\mathbf{r}, t)], \tag{4.1}
 \end{aligned}$$

where

$$\begin{aligned}
 V\Theta_1(\mathbf{r}, t) &= \int_0^\infty d\tau \langle \tau^{+d}(\tau) Y^d \rangle_H(\mathbf{r}, t), \\
 V\Theta_2(\mathbf{r}, t) &= \int_0^\infty d\tau \langle X^d(\tau) \tau^{+d} \rangle_H(\mathbf{r}, t), \tag{4.2} \\
 V\Lambda_1(\mathbf{r}, t) &= \left\langle \sum_j \sum_{k \neq j} \gamma_{jk} \right\rangle_H(\mathbf{r}, t), \quad V\Lambda_2(\mathbf{r}, t) = \int_0^\infty d\tau \langle X^d(\tau) Y^d \rangle_H(\mathbf{r}, t),
 \end{aligned}$$

and  $i$  and  $j$  are summed from 1 to 3. If we define the linear deviations of the hydrodynamic variables to be

$$\begin{aligned}
 \delta n(\mathbf{r}, t) &= n(\mathbf{r}, t) - n_0, \quad \delta \mathbf{v}(\mathbf{r}, t) = \mathbf{v}(\mathbf{r}, t) - \mathbf{v}_0 = \mathbf{v}(\mathbf{r}, t), \\
 \delta e(\mathbf{r}, t) &= e^+(\mathbf{r}, t) - e_0^+, \tag{4.3}
 \end{aligned}$$

where  $n_0$  and  $e_0^+$  are the equilibrium number and internal energy densities respectively, then to linear order in the deviations we obtain

$$\begin{aligned}
 \dot{\delta n}(\mathbf{r}, t) &= -n_0 \nabla_r \cdot \delta \mathbf{v}(\mathbf{r}, t), \\
 mn_0 \dot{\delta \mathbf{v}}(\mathbf{r}, t) &= -a \nabla_r \delta n(\mathbf{r}, t) - b \nabla_r \delta e(\mathbf{r}, t) + (\zeta + \frac{1}{3}\eta) \nabla_r \nabla_r \cdot \delta \mathbf{v}(\mathbf{r}, t) \\
 &\quad + \eta \nabla^2 \delta \mathbf{v}(\mathbf{r}, t), \tag{4.4} \\
 \dot{\delta e}(\mathbf{r}, t) &= -c \nabla_r \cdot \delta \mathbf{v}(\mathbf{r}, t) + \kappa \left( \frac{\partial T}{\partial n} \right)_{e^+} \nabla_r \cdot \nabla_r \delta n(\mathbf{r}, t) \\
 &\quad + \kappa \left( \frac{\partial T}{\partial e^+} \right)_n \nabla_r \cdot \nabla_r \delta e(\mathbf{r}, t) - f \delta n(\mathbf{r}, t) - g \delta e(\mathbf{r}, t),
 \end{aligned}$$

where

$$\begin{aligned}
 a &\equiv \left[ \chi_n - \frac{\lambda \Theta_1}{k_B T^2} \left( \frac{\partial T}{\partial n} \right)_{e^+} \right], \quad b \equiv \left[ \chi_e - \frac{\lambda \Theta_1}{k_B T^2} \left( \frac{\partial T}{\partial e^+} \right)_n \right], \\
 c &\equiv h^+ - \frac{\lambda \Theta_2}{k_B T},
 \end{aligned}$$

$$\begin{aligned}
 f &\equiv \left[ \frac{\lambda \Lambda_1}{mT} \left( \frac{\partial T}{\partial n} \right)_{e^+} - \frac{\lambda^2 \Lambda_2}{k_B T^2} \left( \frac{\partial T}{\partial n} \right)_{e^+} \right], \\
 g &\equiv \left[ \frac{\lambda \Lambda_1}{mT} \left( \frac{\partial T}{\partial e^+} \right)_n - \frac{\lambda^2 \Lambda_2}{k_B T^2} \left( \frac{\partial T}{\partial e^+} \right)_n \right],
 \end{aligned} \tag{4.5}$$

and

$$\chi_n = \left( \frac{\partial P}{\partial n} \right)_{e^+}, \quad \chi_e = \left( \frac{\partial P}{\partial e^+} \right)_n.$$

These equations may be written in Fourier space as

$$\begin{aligned}
 \dot{\delta n}(k, t) &= i n_0 k \delta v_e(k, t), \\
 \dot{\delta e}(k, t) &= i c k \delta v_e(k, t) - \left[ \kappa \left( \frac{\partial T}{\partial n} \right)_{e^+} k^2 + f \right] \delta n(k, t) \\
 &\quad - \left[ \kappa \left( \frac{\partial T}{\partial e^+} \right)_n k^2 + g \right] \delta e(k, t),
 \end{aligned} \tag{4.6}$$

$$\dot{\delta v}_e = \frac{i a k}{m n_0} \delta n(k, t) + \frac{i b k}{m n_0} \delta e(k, t) - \left( \zeta + \frac{4}{3} \eta \right) \frac{k^2}{m n_0} \delta v_e(k, t),$$

$$\dot{\delta v}_t(k, t) = - \frac{\eta}{m n_0} k^2 \delta v_t(k, t),$$

where we have defined the longitudinal and transverse velocities by

$$\delta v_e(k, t) = \hat{k} \cdot \delta \mathbf{v}(k, t), \quad \delta v_t(k, t) = \hat{k} \wedge \delta \mathbf{v}(k, t). \tag{4.7}$$

Note that the linearized equation of motion for the transverse velocity  $\delta \mathbf{v}_t(\mathbf{r}, t)$  is uncoupled from the other hydrodynamic densities and is identical to that obtained for ordinary hydrodynamic systems. From eq. (4.6), we conclude that

$$\delta \mathbf{v}_t(k, t) = \exp\left(- \frac{\eta}{m n_0} k^2 t\right) \delta \mathbf{v}_t(k). \tag{4.8}$$

If we set  $\lambda$  to zero in eq. (4.6), the ordinary hydrodynamic equations are obtained. The eigenmodes or hydrodynamic modes of this system of equations are given to lowest order in  $k$  by

$$\begin{aligned}
 T(k) &= \delta e(k) - \frac{h^+}{n_0} \delta n(k), \\
 \sigma_1(k) &= \frac{\chi_n}{c_0} \delta n(k) + \frac{\chi_e}{c_0} \delta e(k) + m \delta v_e(k), \\
 \sigma_2(k) &= \frac{\chi_n}{c_0} \delta n(k) + \frac{\chi_e}{c_0} \delta e(k) - m \delta v_e(k),
 \end{aligned} \tag{4.9}$$

where  $c_0$  is the zero frequency adiabatic sound velocity defined by

$$c_0^2 = \frac{\chi_n}{m} + \frac{\chi_e h^+}{mn_0}. \tag{4.10}$$

The time dependence of the hydrodynamic modes is given by

$$\begin{aligned} T(k, t) &= \exp(-\Gamma_T k^2 t) T(k), \\ \sigma_1(k, t) &= \exp[(ic_0 k - \Gamma_S k^2)t] \sigma_1(k), \\ \sigma_2(k, t) &= \exp[(-ic_0 k - \Gamma_S k^2)t] \sigma_2(k), \end{aligned} \tag{4.11}$$

where the sound attenuation and thermal diffusivity coefficients are

$$\begin{aligned} \Gamma_S &= \frac{1}{2} \left[ \lambda_H \left( \frac{\partial T}{\partial e^+} \right)_n + \frac{\zeta + \frac{4}{3}\eta}{mn_0} - \Gamma_T \right], \\ \Gamma_T &= \frac{\lambda_H}{mc_0^2} \left[ \chi_n \left( \frac{\partial T}{\partial e^+} \right)_n - \chi_e \left( \frac{\partial T}{\partial n} \right)_{e^+} \right], \end{aligned} \tag{4.12}$$

and  $\lambda_H \equiv \int_0^\infty d\tau \frac{1}{3} \langle \mathbf{J}_E^D(\tau) \cdot \mathbf{J}_E^D \rangle$ .

If we set  $k = 0$  in (4.6), we see that

$$\dot{\delta n}(t) = \delta \dot{v}_e(t) = 0, \quad \dot{\delta e}(t) = -f \delta n(t) - g \delta e(t), \tag{4.13}$$

which implies that  $\delta e(t)$  is an eigenmode with eigenvalue

$$z_1 = \left( \frac{\partial T}{\partial e^+} \right)_n \left( \frac{-\Lambda_1 \lambda}{mT} + \frac{\Lambda_2 \lambda^2}{k_B T^2} \right). \tag{4.14}$$

Note that since  $\lambda \ll 1$ , the energy mode decays to zero in uniform systems.

The general solution to the system of equations in (4.6) for nonzero  $\lambda$  and  $k$  are given in the appendix. The general eigenvalues are given by complicated expressions involving cube roots and suggest nonanalytic  $k$  and  $\lambda$  behavior from which it is difficult to gain any insight, particularly since there are new transport coefficients involved. We therefore consider the limiting cases in which  $\lambda \sim k^2$  so that  $\lambda \ll k$ , and  $\lambda^2 \sim k$  so that  $\lambda \gg k$ .

When  $\lambda \sim k^2$  we find that the hydrodynamic modes are unchanged from the  $\lambda = 0$  case given in eq. (4.9) to lowest order in  $k$ . Furthermore, the real part of the hydrodynamic roots are shifted from  $\Gamma_T$  to  $\tilde{\Gamma}_T$  and from  $\Gamma_S$  to  $\tilde{\Gamma}_S$  where

$$\begin{aligned}\tilde{\Gamma}_T &= \Gamma_T + \frac{1}{mc_0^2} \left( \frac{\lambda}{k^2} \right) \left[ \chi_n \left( \frac{\partial T}{\partial e^+} \right)_n - \chi_e \left( \frac{\partial T}{\partial n} \right)_{e^+} \right] \frac{\Lambda_1}{mT}, \\ \tilde{\Gamma}_S &= \frac{1}{2} \left[ \lambda_H \left( \frac{\partial T}{\partial e^+} \right)_n + \frac{\zeta + \frac{4}{3}\eta}{mn_0} - \tilde{\Gamma}_T \right].\end{aligned}\quad (4.15)$$

When  $\lambda^2 \sim k$ , on the other hand, we find the eigenvalue roots become

$$\begin{aligned}z_1 &= \left( \frac{\partial T}{\partial e^+} \right)_n \left( \frac{-\Lambda_1 \lambda}{mT} + \frac{\Lambda_2 \lambda^2}{k_B T^2} \right) + \left( \frac{k^2}{\lambda^4} \right) \frac{mT \lambda^3}{3\Lambda_1} \left( \frac{\partial e^+}{\partial T} \right)_n (c^2 + c_0^2) + \mathcal{O}(\lambda^4), \\ z_2 &= ic \left( \frac{k}{\lambda^2} \right) \lambda^2 - \frac{mT}{2\Lambda_1} \left( \frac{\partial e^+}{\partial T} \right)_n (c^2 + c_0^2) \left( \frac{k}{\lambda^2} \right)^2 \lambda^3 + \mathcal{O}(\lambda^4), \\ z_3 &= -ic \left( \frac{k}{\lambda^2} \right) \lambda^2 - \frac{mT}{2\Lambda_1} \left( \frac{\partial e^+}{\partial T} \right)_n (c^2 + c_0^2) \left( \frac{k}{\lambda^2} \right)^2 \lambda^3 + \mathcal{O}(\lambda^4),\end{aligned}\quad (4.16)$$

where

$$c = \left( \frac{\chi_n}{m} - \frac{\chi_e}{m} \left( \frac{\partial T / \partial n}{\partial T / \partial e^+} \right)_n \right)^{1/2}, \quad (4.17)$$

and the definitions of  $\Gamma$  and  $\gamma$  are given in (4.2). It can be established from the stability conditions for equilibrium that  $c^2 > 0$  and hence two propagating modes and one diffusive mode exist when  $\lambda^2 \sim k$ . These modes are analogous to the sound and heat modes that are observed when  $\lambda \sim k^2$  and when  $\lambda = 0$ . Note that as  $k$  approaches zero roots  $z_2$  and  $z_3$  approach zero and  $z_1$  approaches the root given in eq. (4.14). It is also worth noting that

$$\Gamma_T = \frac{c^2}{mc_0^2} \lambda_H \left( \frac{\partial T}{\partial e^+} \right)_n. \quad (4.18)$$

## 5. Summary and conclusions

In this paper we have derived generalized Fokker–Planck and Smoluchowski equations for the translational distribution functions. The Smoluchowski equation leads to clustering in configuration space. We have also derived the nonlinear hydrodynamic equations for a system of  $N$  particles in which energy gradually flows unidirectionally from translational degrees of freedom into internal modes of the particles during collisions. A small parameter  $\epsilon$  appears in the effective Liouvillian for the evolution of the translational degrees of freedom and is associated with the flow of energy into internal modes. Terms

containing  $\gamma_{jk}/m$  are small when averaged over the translational degrees of freedom. We have therefore associated a small parameter  $\lambda$  to each factor of  $\gamma_{jk}/m$  that appears in the hydrodynamic equations. We have limited our analysis to the case in which the energy density is slowly varying over the timescale  $\tau_m$  and have therefore restricted our analysis to the regime in which the gradients of the system and  $\lambda$  are small.

The resulting nonlinear hydrodynamic equations for granular flow systems to second order in gradients and  $\lambda$  are given in eq. (4.1). These equations have additional terms in the equations of motion for the energy and momentum densities which are not present in ordinary hydrodynamics. The linearized granular flow equations, given in eq. (4.4), have many features in common with the ordinary linearized equations, which can be obtained from equation (4.4) by setting  $\lambda = 0$ . When  $\lambda \sim k^2$ , where  $k$  is the magnitude of the wavevector of the system, the additional terms which appear in the momentum and energy density equations change the damping rates of the thermal and sound modes from  $\Gamma_T$  to  $\tilde{\Gamma}_T$  and  $\Gamma_S$  to  $\tilde{\Gamma}_S$  (see eq. (4.15)). When  $\lambda^2 \sim k$ , there are two counter-propagating modes with speed

$$c \equiv \left( \frac{\chi_n}{m} - \frac{\chi_e}{m} \frac{(\partial T / \partial n)_{e^+}}{(\partial T / \partial e^+)_{n^+}} \right)^{1/2} \tag{5.1}$$

and a diffusive mode with an eigenvalue root of

$$z_1 = \left( \frac{\partial T}{\partial e^+} \right)_n \left( \frac{-\Lambda_1 \lambda}{mT} + \frac{\Lambda_2 \lambda^2}{k_B T^2} \right) + \mathcal{O}(\lambda^3). \tag{5.2}$$

As  $k$  approaches zero, only the diffusive root remains nonzero, which reflects the fact that energy is no longer conserved even though the momentum and total number of particles are.

The task of generalizing the nonlinear hydrodynamic equations for granular flow systems to higher order in  $\lambda$  is a daunting task since the effective Liouvillian operator  $O^\dagger$  must be generalized to include higher powers of  $\epsilon$  (see eq. (2.28)) to consistently calculate the higher order terms of  $O^\dagger A$  and  $\psi(t)$  which appear in the equation of motion for the densities which compose the set  $A$  in the equation

$$\dot{a}(\mathbf{r}, t) = \langle O^\dagger A \rangle_t + \int_0^\infty d\tau \langle (e^{\mathcal{L}(\tau)} O^\dagger \mathcal{Q}(\tau) O^\dagger A) \mathcal{Q}(\tau) \psi(\tau) \rangle_t. \tag{5.3}$$

In addition to these difficulties, the mode coupling contributions to the granular flow hydrodynamic equations which have been ignored in this paper

will introduce nonanalytic powers of  $\lambda$  and  $k$  just as they do in the ordinary hydrodynamic equations [11]. Based upon computer simulations [12], mode coupling seems to play a very important role in granular flow dynamics since the flow of energy into internal modes of the particles enhances clustering effects and coordinated flow of the granular particles. The effect of mode coupling on the hydrodynamic equations for granular flow systems is currently under investigation and will be presented in a future publication.

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### Appendix

In this appendix, we examine the eigenvalues of the matrix

$$\begin{pmatrix} 0 & in_0k & 0 \\ \frac{i\chi_n k}{mn_0} - ia\lambda\kappa & -\frac{\nu_e k^2}{mn_0} & \frac{i\chi_e k}{mn_0} - ib\lambda k \\ -c\lambda + d\lambda^2 - \kappa_n k^2 & ih^+k - if\lambda k & -g\lambda + h\lambda^2 - \kappa_e k^2 \end{pmatrix}, \quad (\text{A.1})$$

where

$$\begin{aligned} a &\equiv \frac{\Theta_1}{k_B T^2 mn_0} \left( \frac{\partial T}{\partial n} \right)_{e^+}, & b &\equiv \frac{\Theta_1}{k_B T^2 mn_0} \left( \frac{\partial T}{\partial e^+} \right)_n, & c &\equiv \frac{\Lambda_1}{mT} \left( \frac{\partial T}{\partial n} \right)_{e^+}, \\ d &\equiv \frac{\Lambda_2}{k_B T^2} \left( \frac{\partial T}{\partial n} \right)_{e^+}, & f &\equiv \frac{\Theta_2}{k_B T}, & g &\equiv \frac{\Lambda_1}{mT} \left( \frac{\partial T}{\partial e^+} \right)_n, & h &\equiv \frac{\Lambda_2}{k_B T^2} \left( \frac{\partial T}{\partial e^+} \right)_n, \\ \kappa_n &\equiv \lambda\chi_n, & \kappa_e &\equiv \lambda\chi_e, & \nu_e &\equiv \zeta + \frac{4}{3}\eta, \end{aligned} \quad (\text{A.2})$$

and  $\Theta_1$ ,  $\Theta_2$ ,  $\Lambda_1$  and  $\Lambda_2$  are defined in (4.1). The eigenvalues of (A.1) are the three roots of the cubic equation

$$\begin{aligned} z^3 + z^2 \left[ g\lambda - h\lambda^2 + \left( \kappa_e + \frac{\nu_e}{mn_0} \right) k^2 \right] \\ + z \left[ \left( \frac{\nu_e g}{mn_0} - \frac{\chi_e f}{mn_0} - h^+b - n_0a \right) k^2 \lambda \right] \end{aligned}$$

$$\begin{aligned}
 & + \left( bf - \frac{v_\ell h}{mn_0} \right) k^2 \lambda^2 + c_0^2 k^2 + \frac{v_\ell \kappa_e}{mn_0} k^4 \Big] \\
 & + \left[ \left( \frac{\chi_n \kappa_e}{m} - \frac{\chi_e \kappa_n}{m} \right) k^4 + \left( \frac{\chi_n g}{m} - \frac{\chi_e c}{m} \right) k^2 \lambda + (n_0 \kappa_n b - n_0 \kappa_e a) k^4 \lambda \right. \\
 & \left. + \left( \frac{\chi_e d}{m} - \frac{\chi_n h}{m} \right) k^2 \lambda^2 \right] \\
 & \equiv z^3 + a_2 z^2 + a_1 z + a_0 = 0.
 \end{aligned} \tag{A.3}$$

The general solution of the cubic equation is

$$\begin{aligned}
 z_1 &= (S_1 + S_2) - \frac{1}{3} a_2, \\
 z_2 &= -\frac{1}{2} (S_1 + S_2) - \frac{1}{3} a_2 + \frac{1}{2} i \sqrt{3} (S_1 - S_2), \\
 z_3 &= -\frac{1}{2} (S_1 + S_2) - \frac{1}{3} a_2 - \frac{1}{2} i \sqrt{3} (S_1 - S_2),
 \end{aligned} \tag{A.4}$$

where

$$S_1 = [r + (q^3 + r^2)^{1/2}]^{1/3}, \quad S_2 = [r - (q^3 + r^2)^{1/2}]^{1/3},$$

and

$$q = \frac{1}{3} a_1 - \frac{1}{9} a_2^2, \quad r = \frac{1}{6} (a_1 a_2 - 3 a_0) - \frac{1}{27} a_2^3.$$

Since we are interested only in the leading powers of  $\lambda$  and  $k$ , these roots may be simplified somewhat. However, due to the nonanalytic nature of the roots, the expansions in powers of  $\lambda$  and  $k$  are cumbersome unless  $\lambda \gg k$  or  $k \gg \lambda$ . The roots in these limits are presented in the main text.

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