Physica A 204 (1994) 555-605
North-Holland

SSDI 0378-4371(93)E0364-K

Mode coupling in nonequilibrium granular flow systems

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In this paper we examine the role of mode coupling in nonequilibrium granular flow systems and derive mode coupling equations for equal-time multi-linear correlation functions in general nonequilibrium systems. By applying N-ordering analysis and projection operator techniques, the hierarchy of equations obtained is simplified which permits the equal-time multilinear averages to be calculated systematically. The linear and equal-time bilinear nonequilibrium correlation functions are calculated explicitly for a steady-state system characterized by a small linear shear flow and uniform number and internal energy densities.

1. Introduction

In this paper we examine the importance of mode coupling [1] in nonequilibrium systems and investigate the properties of equal-time correlations of multilinear densities in steady-state granular systems. In recent years, there has been intense study of the transport properties of granular fluid systems composed of sand or glass beads. Most of these studies have been semi-phenomenological and involve the concept of the coefficient of restitution, which is a measure of the inelasticity of collisions among the granular particles composing the fluid. As examples of these studies, we mention the analytical treatments of Jenkins et al. [2] and the simulations by Walton et al. [3]. The inelasticity is essential for the treatment of these systems and for comparisons with experimental data [4].

The paper is organized as follows: Using ideas developed to describe Brownian motion dynamics [5], in section 2 we obtain a generalized Fokker–Planck equation for the distribution function of the translational degrees of freedom of the granular system and thereby derive an effective time-displacement operator which determines the time evolution of the distribution function. In section 3 we construct a mode coupling theory for a general system whose dynamics are determined by a time-displacement operator O by considering a generalization of the local equilibrium distribution to describe the nonequilibrium state which independently constrains all the moments of the hydrodynamic variables. Using the N-ordering scheme developed in earlier work [6,7], the infinite hierarchy of mode coupling equations for the moments of the hydrodynamic variables in the nonequilibrium system is truncated and an equation is obtained which permits the generalized thermodynamic forces $\phi^{(n)}(t)$ of the generalized local equilibrium distribution to be calculated systematically to leading N order and leading wavevector. These forces, in turn, allow the calculation of the leading wavevector and N order dependence of the equal-time correlations of the multilinear moments of the hydrodynamic variables. In section 4 we rederive the mode coupling equations for equal-time multilinear correlations in nonequilibrium systems following an alternative approach which gives insight into the space and time dependent nonequilibrium averages. In section 5, using the effective time-displacement operator O derived in section 2, we analyze the mode coupling equations for a granular flow system. Finally, in section 6 we examine the mode coupling expressions for bilinear variables and calculate equal-time correlation functions for a nonequilibrium steady-state granular flow system which is characterized by linear shear flow and uniform number and internal energy densities.

We expect the mode coupling equations we will derive to apply to granular systems since energy flows unidirectionally from the translational degrees of freedom into internal modes of the particles in the collisions between granular particles. Each granular particle contains many molecules and internal modes which can be modelled as a bath with an internal temperature, T, which essentially remains in equilibrium throughout collisions. Note that when the translational temperature is T, the center of mass velocities of the particles are essentially zero.

2. The granular flow model and generalized Fokker–Planck equations

The granular system consists of N identical spherical particles whose center of mass coordinates and momenta are denoted by $\boldsymbol{r}^N$ and $\boldsymbol{p}^N$ respectively, where $\boldsymbol{r}^N$ and $\boldsymbol{p}^N$ are $3N$-dimensional vectors with components $\boldsymbol{r}_j$ and $\boldsymbol{p}_j$ with $j = 1, \ldots, N$. Each of the N granular particles contains many internal modes. We will denote the internal coordinates of particle j by the vector $\boldsymbol{\xi}_j$, and the internal momenta of particle j by $\boldsymbol{\pi}_j$. We assume that the number of internal momenta and coordinates of each particle is sufficiently large and the interactions among them sufficiently strong that their distribution function is of equilibrium form. We denote the phase point for the translational degrees of freedom by $X_\mathrm{t} \equiv (\boldsymbol{r}^N, \boldsymbol{p}^N)$ and the phase point for the internal degrees of freedom by $X_\mathrm{i} \equiv (\boldsymbol{\xi}^N, \boldsymbol{\pi}^N)$.

The Hamiltonian for the granular flow system can be written as

$$H(X_t, X_i) = H_t(X_t) + H_i(X_i) + \phi(X_t, X_i) , \tag{2.1}$$

where H_t is the Hamiltonian for the isolated translational degrees of freedom, H_i is the Hamiltonian for the isolated internal degrees of freedom and ϕ describes the interactions between the translational and internal degrees of freedom. The Hamiltonian H_t has the form

$$H_t(X_t) = \frac{\boldsymbol{p}^N \cdot \boldsymbol{p}^N}{2m} + U(\boldsymbol{r}^N) , \tag{2.2}$$

where m is the mass of the granular particle and U can be written as a sum of two-body short-ranged potentials,

$$U(\boldsymbol{r}^N) = \sum_{i=1}^{N} \sum_{j>i} U(r_{ij}) , \tag{2.3}$$

where $r_{ij} \equiv |\boldsymbol{r}_i - \boldsymbol{r}_j|$. The Hamiltonian H_i is of the form

$$H_i = \sum_{j=1}^{N} \left(\frac{\boldsymbol{\pi}_j \cdot \boldsymbol{\pi}_j}{2\mu} + V(\boldsymbol{\xi}_j) \right) . \tag{2.4}$$

The interaction term ϕ can be written as a sum of two-body terms, each of which depends on the distance between the center of mass of a pair of particles and on the internal coordinates of that pair,

$$\phi(X_t, X_i) = \sum_{i=1}^{N} \sum_{j>i} \phi(r_{ij}, \boldsymbol{\xi}_i, \boldsymbol{\xi}_j) . \tag{2.5}$$

We assume that ϕ is short-ranged and decays to zero for $r_{ij} > \sigma$, where σ represents the range of interaction, which we expect to be of the order of the diameter of a particle.

The Liouvillian operator for the system can be written as

$$L = L_t + L_i + L_\phi , \tag{2.6}$$

where

$$\begin{aligned} L_t &= -\frac{\boldsymbol{p}^N}{m} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}^N} + \boldsymbol{\nabla}_{\boldsymbol{r}^N} U \cdot \boldsymbol{\nabla}_{\boldsymbol{p}^N} , \\ L_i &= -\frac{\boldsymbol{\pi}^N}{\mu} \cdot \boldsymbol{\nabla}_{\boldsymbol{\xi}^N} + \boldsymbol{\nabla}_{\boldsymbol{\xi}^N} V \cdot \boldsymbol{\nabla}_{\boldsymbol{\pi}^N} , \end{aligned} \tag{2.7}$$

and

$$L_\phi = \nabla_{r^N}\phi \cdot \nabla_{p^N} + \nabla_{\xi^N}\phi \cdot \nabla_{\pi^N} . \tag{2.8}$$

The Liouville operator L determines the dynamics of the total distribution function $\rho(X_t, X_i, t)$ of the granular flow system,

$$\dot{\rho}(t) = L\rho(t) . \tag{2.9}$$

We shall use a projection operator technique [8] to obtain an equation of motion for the reduced distribution function of the translational degrees of freedom $W(X_t, t)$, where

$$W(X_t, t) = \int dX_i \, \rho(X_t, X_i, t) . \tag{2.10}$$

At total equilibrium, the distribution functions are

$$\rho_e = \frac{e^{-\beta H_t} e^{-\beta H_i} e^{-\beta\phi}}{q} , \tag{2.11}$$

and

$$W_e = \frac{e^{-\beta H_t} \int dX_i \, e^{-\beta H_i} e^{-\beta\phi}}{q} , \tag{2.12}$$

where

$$q \equiv \int dX_t \, dX_i \, e^{-\beta H_t} e^{-\beta H_i} e^{-\beta\phi} ,$$

$\beta \equiv (K_B T)^{-1}$, and T is the equilibrium temperature of the overall system. It will be useful to introduce the equilibrium distribution function for the isolated internal degrees of freedom,

$$\rho_i(X_i) \equiv \frac{e^{-\beta H_i}}{\int dX_i \, e^{-\beta H_i}} , \tag{2.13}$$

and define the potential of mean force, $\omega(\boldsymbol{r}^N)$, by

$$\exp[-\beta\omega(\boldsymbol{r}^N)] \equiv \int dX_i \, \rho_i \, e^{-\beta\phi} \equiv \langle e^{-\beta\phi} \rangle_0 . \tag{2.14}$$

Following standard projection operator techniques [9], we find that

$$\begin{aligned}\dot{W}(t) &\approx \Big(-\frac{\boldsymbol{p}^N}{m} \cdot \boldsymbol{\nabla}_{\boldsymbol{p}^N} + \boldsymbol{\nabla}_{\boldsymbol{r}^N}(U + \omega) \cdot \boldsymbol{\nabla}_{\boldsymbol{p}^N} + \Omega_1 + \Omega_2 \Big) W(t) \\ &\equiv O W(t) ,\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}\Omega_1 &= \tfrac{1}{2} \sum_j \sum_{k \neq j} \gamma_{jk} \hat{\boldsymbol{r}}_{jk} \hat{\boldsymbol{r}}_{jk} : \Big(\frac{\beta}{m} (\boldsymbol{\nabla}_{\boldsymbol{p}_j} - \boldsymbol{\nabla}_{\boldsymbol{p}_k})(\boldsymbol{p}_j - \boldsymbol{p}_k) \Big) , \\ \Omega_2 &= \tfrac{1}{2} \sum_j \sum_{k \neq j} \gamma_{jk} \hat{\boldsymbol{r}}_{jk} \hat{\boldsymbol{r}}_{jk} : [(\boldsymbol{\nabla}_{\boldsymbol{p}_j} - \boldsymbol{\nabla}_{\boldsymbol{p}_k})(\boldsymbol{\nabla}_{\boldsymbol{p}_j} - \boldsymbol{\nabla}_{\boldsymbol{p}_k})]\end{aligned} \tag{2.16}$$

and

$$\gamma_{jk} \hat{\boldsymbol{r}}_{jk} \hat{\boldsymbol{r}}_{jk} \equiv \int_0^\infty \mathrm{d}\tau \, \langle \widehat{\boldsymbol{\nabla}_{\boldsymbol{r}_{jk}}} \phi \{ \exp[(L_\mathrm{i} + \boldsymbol{\nabla}_{\boldsymbol{\xi}^N} \phi \cdot \boldsymbol{\nabla}_{\boldsymbol{\pi}^N}) \tau] \, \widehat{\boldsymbol{\nabla}_{\boldsymbol{r}_{jk}}} \phi \} \rangle_\mathrm{i} , \tag{2.17}$$

with

$$\begin{aligned}\langle B \rangle_\mathrm{i} &\equiv \int \mathrm{d}X_\mathrm{i} \, \mathrm{e}^{-\beta\phi} \, \mathrm{e}^{\beta\omega} \rho_\mathrm{i} B , \\ \widehat{\boldsymbol{\nabla}_{\boldsymbol{r}_{jk}}} \phi &\equiv \boldsymbol{\nabla}_{\boldsymbol{r}_{jk}} (\phi - \omega) .\end{aligned} \tag{2.18}$$

Eq. (2.15) defines the effective time-displacement operator O for the granular flow distribution function $W(t)$. The operator O is not a first order differential operator nor is it Hermitian (or anti-Hermitian). In later sections we shall apply projection operator techniques to eq. (2.15) to obtain dynamical equations for nonequilibrium averages of densities of the system. Kawasaki [10] has considered similar situations in a recent paper.

It can be shown that any arbitrary function $G(X_\mathrm{t}, t)$ of the translational phase space $X_\mathrm{t} \equiv (\boldsymbol{r}^N, \boldsymbol{p}^N)$ obeys the generalized Langevin equation,

$$\dot{G}(X_\mathrm{t}, t) = K_G(t) + O^\dagger G(X_\mathrm{t}, t) , \tag{2.19}$$

where

$$O^\dagger = \Big(\frac{\boldsymbol{p}^N}{m} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}^N} - \boldsymbol{\nabla}_{\boldsymbol{r}^N}(U + \omega) \cdot \boldsymbol{\nabla}_{\boldsymbol{p}^N} \Big) + \Omega_1^\dagger + \Omega_2 , \tag{2.20}$$

$$\Omega_1^\dagger = -\tfrac{1}{2} \sum_j \sum_{k \neq j} \gamma_{jk} \hat{\boldsymbol{r}}_{jk} \hat{\boldsymbol{r}}_{jk} : \Big(\frac{\beta}{m} (\boldsymbol{p}_j - \boldsymbol{p}_k)(\boldsymbol{\nabla}_{\boldsymbol{p}_j} - \boldsymbol{\nabla}_{\boldsymbol{p}_k}) \Big)$$

and

$$K_G(t) \equiv -\mathrm{e}^{-(1-\tilde{\mathcal{P}})Lt}(1-\tilde{\mathcal{P}})LG\ ,$$

with $\tilde{\mathcal{P}}B \equiv \langle B\rangle_{\mathrm{i}}$. Note that due to the terms proportional to Ω_1 in O, $O \neq -O^{\dagger}$. It will prove to be convenient to introduce the reduced momentum

$$\boldsymbol{p}_j^+ \equiv \boldsymbol{p}_j - m\boldsymbol{v}(\boldsymbol{r}_j) \tag{2.21}$$

and the reduced momentum density

$$\boldsymbol{P}^+(r) \equiv \sum_j \boldsymbol{p}_j^+ \delta(\boldsymbol{r} - \boldsymbol{r}_j)\ . \tag{2.22}$$

We then find that

$$\begin{aligned} \boldsymbol{P}(\boldsymbol{r}) &= \boldsymbol{P}^+(\boldsymbol{r}) + m\boldsymbol{v}(\boldsymbol{r})\, N(\boldsymbol{r})\ ,\\ E(\boldsymbol{r}) &= E^+(\boldsymbol{r}) + \boldsymbol{v}(\boldsymbol{r})\cdot\boldsymbol{P}^+(\boldsymbol{r}) + \tfrac{1}{2}mv^2(\boldsymbol{r})\, N(\boldsymbol{r})\ , \end{aligned} \tag{2.23}$$

where the dynamical variable $E^+(\boldsymbol{r})$ is the internal energy density of the system.

The nonlinear operator Ω_2 operating on $A(\boldsymbol{r}_1)A(\boldsymbol{r}_2)$ yields

$$\Omega_2[A(\boldsymbol{r}_1)A(\boldsymbol{r}_2)] = [\Omega_2 A(\boldsymbol{r}_1)]A(\boldsymbol{r}_2) + A(\boldsymbol{r}_1)\,[\Omega_2 A(\boldsymbol{r}_2)] + \Omega_2'[A(\boldsymbol{r}_1)\,A(\boldsymbol{r}_2)]\ , \tag{2.24}$$

where

$$\Omega_2'[A(\boldsymbol{r}_1)\,A(\boldsymbol{r}_2)] \equiv \sum_j \sum_{k\neq j} \gamma_{jk}\hat{\boldsymbol{r}}_{jk}\hat{\boldsymbol{r}}_{jk} : [(\boldsymbol{\nabla}_{\boldsymbol{p}_j} - \boldsymbol{\nabla}_{\boldsymbol{p}_k})A(\boldsymbol{r}_1)\,(\boldsymbol{\nabla}_{\boldsymbol{p}_j} - \boldsymbol{\nabla}_{\boldsymbol{p}_k})A(\boldsymbol{r}_2)]\ . \tag{2.25}$$

Explicit evaluation of the Ω_2' terms reveals that

$$\begin{aligned} \Omega_2'[N(\boldsymbol{r}_1)\,A(\boldsymbol{r}_2)] &= 0\ ,\\ \Omega_2'[\boldsymbol{P}^+(\boldsymbol{r}_1)\,\boldsymbol{P}^+(\boldsymbol{r}_2)] &= 2\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\sum_j\sum_{k\neq j}\gamma_{jk}\hat{\boldsymbol{r}}'_{jk}\hat{\boldsymbol{r}}'_{jk}\delta(\boldsymbol{r}'_j - \boldsymbol{r}_1)\\ &\quad - 2\gamma_{12}(\boldsymbol{r}_{12})\,\hat{\boldsymbol{r}}_{12}\hat{\boldsymbol{r}}_{12}N(\boldsymbol{r}_1)\,N(\boldsymbol{r}_2)\ ,\\ \Omega_2'[\boldsymbol{P}^+(\boldsymbol{r}_1)\,E^+(\boldsymbol{r}_2)] &= 2\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\sum_j\sum_{k\neq j}\frac{\gamma_{jk}}{m}\hat{\boldsymbol{r}}'_{jk}\hat{\boldsymbol{r}}'_{jk}\cdot\boldsymbol{P}_j^+\delta(\boldsymbol{r}'_j - \boldsymbol{r}_1)\\ &\quad - \frac{2}{m}\gamma_{12}(\boldsymbol{r}_{12})\,\hat{\boldsymbol{r}}_{12}\hat{\boldsymbol{r}}_{12}\cdot N(\boldsymbol{r}_1)\,\boldsymbol{P}^+(\boldsymbol{r}_2)\ , \end{aligned}$$

$$\Omega_2'[E^+(\boldsymbol{r}_1)\,E^+(\boldsymbol{r}_2)] = 2\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) \sum_j \sum_{k \neq j} \frac{\gamma_{jk}}{m^2}\, \hat{\boldsymbol{r}}'_{jk}\hat{\boldsymbol{r}}'_{jk} : \boldsymbol{P}_j^+ \boldsymbol{P}_j^+ \delta(\boldsymbol{r}_j' - \boldsymbol{r}_1)$$
$$- \frac{2}{m^2}\gamma_{12}(\boldsymbol{r}_{12})\, \hat{\boldsymbol{r}}_{12}\hat{\boldsymbol{r}}_{12} : \boldsymbol{P}^+(\boldsymbol{r}_1)\, \boldsymbol{P}^+(\boldsymbol{r}_2)\,. \tag{2.26}$$

It is important to note that the terms due to the nonlinearity of the operator O involving Ω_2' yield short-ranged functions of $r_{12} \equiv |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ when operating on $A(\boldsymbol{r}_1)\, A(\boldsymbol{r}_2)$.

3. The mode coupling formalism

In this section, we construct the formal mode coupling formalism for a general system which will enable us to calculate the leading wavevector dependence of multilinear equal-time correlations of the hydrodynamic densities. We shall generalize the projection operator methods of Levine and Oppenheim [11] to construct the formalism. We therefore define a special set of variables $A(\boldsymbol{r}, X_t) \equiv A(\boldsymbol{r})$ to be a vector whose components are the number density $N(\boldsymbol{r})$, the energy density $E(\boldsymbol{r})$ and the momentum density $\boldsymbol{P}(\boldsymbol{r})$. We shall denote the average of an arbitrary function $G(\boldsymbol{r})$ of the phase point X_t over the grand canonical equilibrium distribution W_e by

$$\langle G(\boldsymbol{r}) \rangle \equiv \int \mathrm{d}X_t\, W_e G(\boldsymbol{r})\,, \tag{3.1}$$

and the nonequilibrium average of $G(\boldsymbol{r})$ by

$$\bar{G}(t) \equiv \int \mathrm{d}X_t\, G(X_t)\, W(t)\,, \tag{3.2}$$

where $W(t)$ is the reduced distribution for the translational degrees of freedom (see eq. (2.10)). We now define the local equilibrium distribution function $\sigma_1(t)$ to be

$$\sigma_1(t) \equiv \frac{\exp[\phi^{(1)}(\boldsymbol{r}, t) * A(\boldsymbol{r})]\, W_e(X_t)}{\int \mathrm{d}X_t\, W_e(X_t) \exp[\phi^{(1)}(\boldsymbol{r}, t) * A(\boldsymbol{r})]}\,, \tag{3.3}$$

where the $*$ product notation implies an integration over the repeated spatial argument $\boldsymbol{r}$ as well as a vector product,

$$\boldsymbol{\phi}^{(1)}(\boldsymbol{r},t) * A(\boldsymbol{r}) \equiv \sum_n \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{\phi}_n^{(1)}(\boldsymbol{r},t)\, A_n(\boldsymbol{r}) . \tag{3.4}$$

The $\boldsymbol{\phi}_n^{(1)}(\boldsymbol{r},t)$ are chosen such that

$$\overline{A(\boldsymbol{r})}(t) \equiv \int \mathrm{d}X_t\, A(\boldsymbol{r})\, W(t) = \int \mathrm{d}X_t\, A(\boldsymbol{r})\, \sigma_1(t) \equiv \langle A(\boldsymbol{r}) \rangle_{\mathrm{L}} , \tag{3.5}$$

where $\langle\ \rangle_{\mathrm{L}}$ denotes the average over the local equilibrium distribution function $\sigma_1(t)$. It follows from eq. (3.5) that the exact values of the nonequilibrium averages of the special variables in $A(\boldsymbol{r})$ at any time t can be obtained from averaging these variables over the local equilibrium distribution function $\sigma_1(t)$, and the $\boldsymbol{\phi}_n^{(1)}(\boldsymbol{r},t)$ can be considered as Lagrange multipliers which fix the averages of the densities in $A(\boldsymbol{r})$ at each point in space $\boldsymbol{r}$, at time t. Explicitly, the forces $\boldsymbol{\phi}^{(1)}(\boldsymbol{r},t)$ conjugate to the hydrodynamic variables in $A(\boldsymbol{r})$ are given by

$$\begin{gathered}\boldsymbol{\phi}_N^{(1)}(\boldsymbol{r},t) = \beta(\boldsymbol{r},t)[\mu(\boldsymbol{r},t) - \tfrac{1}{2}mv^2(\boldsymbol{r},t)] - \beta\mu , \\ \boldsymbol{\phi}_E^{(1)}(\boldsymbol{r},t) = \beta - \beta(\boldsymbol{r},t) , \qquad \boldsymbol{\phi}_P^{(1)}(\boldsymbol{r},t) = \beta(\boldsymbol{r},t)\, \boldsymbol{v}(\boldsymbol{r},t) ,\end{gathered} \tag{3.6}$$

where $\beta(\boldsymbol{r},t) = [K_{\mathrm{B}}T(\boldsymbol{r},t)]^{-1}$, $\mu(\boldsymbol{r},t)$ and $\boldsymbol{v}(\boldsymbol{r},t)$ are local quantities. We can rewrite the local equilibrium distribution function in eq. (3.3) by substituting (3.6) into (3.3) to obtain

$$\sigma_1(t) = \frac{1}{N!h^{3N}\Xi_{\mathrm{L}}} \exp\Big(- \int \mathrm{d}\boldsymbol{r}\, \beta(\boldsymbol{r},t)\, [E(\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{r},t)\cdot \boldsymbol{P}(\boldsymbol{r}) - \varphi(\boldsymbol{r},t)\, N(\boldsymbol{r})]\Big) , \tag{3.7}$$

where

$$\varphi(\boldsymbol{r},t) \equiv \mu(\boldsymbol{r},t) - \tfrac{1}{2}mv^2(\boldsymbol{r},t) \tag{3.8}$$

and Ξ_{L} is a normalization constant. Using the internal momentum and energy densities, we may rewrite (3.7) as

$$A(\boldsymbol{r}) * \boldsymbol{\phi}^{(1)}(\boldsymbol{r},t) = - \int \mathrm{d}\boldsymbol{r}\, [E^{+}(\boldsymbol{r})\, \beta(\boldsymbol{r},t) - N(\boldsymbol{r})\, \beta(\boldsymbol{r},t)\, \mu(\boldsymbol{r},t)] .$$

We now define the multilinear basis set Q for the nonequilibrium system to be

$$
\begin{aligned}
&Q_0 = 1\,,\\
&Q_1(\boldsymbol{r}_1) = A(\boldsymbol{r}_1) - \langle A(\boldsymbol{r}_1)\rangle_{\mathrm{L}}\,,\\
&Q_1^{(0)}(\boldsymbol{r}_1) = A(\boldsymbol{r}_1) - \langle A(\boldsymbol{r}_1)\rangle \equiv \hat{A}(\boldsymbol{r}_1)\,,\\
&Q_2(\boldsymbol{r}_1,\boldsymbol{r}_2) = Q_1(\boldsymbol{r}_1)\,Q_1(\boldsymbol{r}_2) - \langle Q_1(\boldsymbol{r}_1)\,Q_1(\boldsymbol{r}_2)\rangle_{\mathrm{L}}\\
&\qquad\qquad - \langle Q_1(\boldsymbol{r}_1)\,Q_1(\boldsymbol{r}_2)\,Q_1(\boldsymbol{r}^1)\rangle_{\mathrm{L}} * K_{11}^{(\mathrm{L})-1}(\boldsymbol{r},\boldsymbol{r}') * Q_1(\boldsymbol{r}')\,,\\
&\quad\vdots\\
&Q_n(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_n) = Q_1(\boldsymbol{r}_1)\,Q_1(\boldsymbol{r}_2)\cdots Q_1(\boldsymbol{r}_n) - \langle Q_1(\boldsymbol{r}_1)\cdots Q_1(\boldsymbol{r}_n)\rangle_{\mathrm{L}}\\
&\qquad\qquad - \sum_{i=1}^{n-1} \langle Q_1(\boldsymbol{r}_1)\cdots Q_1(\boldsymbol{r}_n)\,Q_i\rangle_{\mathrm{L}} * K_{i\hat{i}}^{(\mathrm{L})-1} * Q_{\hat{i}}\,,
\end{aligned}
\tag{3.9}
$$

etc., where $K_{i\hat{i}}^{(\mathrm{L})} \equiv \langle Q_i Q_{\hat{i}}\rangle_{\mathrm{L}}$. For simplicity for notation we have suppressed the spatial arguments of the Q_i in eq. (3.9). Here $*$ implies an integration of repeated spatial arguments over the volume of the system and a sum over hydrodynamic labels of the repeated multilinear variables subject to the restriction that each multilinear variable is counted only once [6]. The subtraction in eq. (3.9) are included to insure that

$$\langle Q_i Q_j\rangle_{\mathrm{L}} = K_{ij}\delta_{|i|,|j|}\,,$$

which implies that the components of Q are orthogonal in mode order. It should be noted that the basis set Q is time dependent due to the averages over the local equilibrium distribution function $\sigma_1(t)$.

We now define the generalized local equilibrium distribution function $\sigma(t)$ to be

$$\sigma(t) = \frac{\mathrm{e}^{\boldsymbol{\phi} * Q} W_{\mathrm{e}}}{\langle \mathrm{e}^{\boldsymbol{\phi} * Q}\rangle}\,, \tag{3.10}$$

where the notation $\boldsymbol{\phi} * Q$ with no explicit mode order subscripts denotes

$$\boldsymbol{\phi} * Q = \sum_{n=1}^{\infty} \int \mathrm{d}\boldsymbol{r}_1 \cdots \mathrm{d}\boldsymbol{r}_n\, \boldsymbol{\phi}^{(n)}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_n,t)\cdot Q_n(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_n)\,. \tag{3.11}$$

The generalized thermodynamic forces $\boldsymbol{\phi}^{(n)}$ are chosen order by order such that

$$\overline{Q_n(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_n)}(t) = \int \mathrm{d}X_{\mathrm{t}}\, W(t) Q_n = \int \mathrm{d}X_{\mathrm{t}}\, Q_n \sigma(t) \equiv \langle Q_n\rangle_t\,, \tag{3.12}$$

where $\langle\ \rangle_t$ indicates the average over the distribution function $\sigma(t)$. It follows from eq. (3.12) that the nonequilibrium average of the basis set Q at any time t can be obtained by averaging Q over the generalized local equilibrium distribution function $\sigma(t)$. The generalized thermodynamic forces $\boldsymbol{\phi}^{(n)}(t)$ can be considered as Lagrange multipliers which fix the averages of the multilinear densities Q at time t.

Since it is often the case that the reduced nonequilibrium distribution functions are intermediate-ranged functions of their relative spatial variables, we shall postulate that the generalized thermodynamic forces are intermediate-ranged functions of their relative spatial arguments so that

$$\boldsymbol{\phi}^{(n)}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n, t) = \boldsymbol{\psi}^{(n)}(\boldsymbol{R}, \boldsymbol{r}_{12}, \ldots, \boldsymbol{r}_{1n}, t) \tag{3.13}$$

$$\boldsymbol{\psi}^{(n)}(\boldsymbol{R}, \boldsymbol{r}_{12}, \ldots, \boldsymbol{r}_{1n}, t) \to 0 \qquad \text{for } |\boldsymbol{r}_{1i}| > \xi\ , \tag{3.14}$$

where $\boldsymbol{R} \equiv (\boldsymbol{r}_1 + \cdots + \boldsymbol{r}_n)/n$ is the center of mass coordinate, $\boldsymbol{r}_{1i} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_i$ are the relative coordinates of $\boldsymbol{\phi}^{(n)}$, and ξ is some intermediate distance characterizing the range of the relative arguments of $\boldsymbol{\psi}^{(n)}$. We expect that ξ is larger than the equilibrium and local equilibrium correlation lengths which we represent by a but smaller than hydrodynamic lengths. We shall demonstrate that this postulate is consistent with the values of $\boldsymbol{\phi}^{(n)}$ obtained for hydrodynamic systems linearly removed from equilibrium by a constant shear flow. We expect the $\boldsymbol{R}$ dependence of the $\boldsymbol{\phi}^{(n)}$ to extend over the whole volume of the system since $\boldsymbol{\phi}^{(1)}$ just refers to the local values of thermodynamic properties, such as the temperature, which are defined over the entire system.

To construct a tractable formalism for the evaluation of equal-time correlations in nonequilibrium systems, we shall apply the N-ordering expansion developed previously for equilibrium systems [6,7] to local equilibrium correlation functions. When calculating N orders, each equilibrium cumulant is assigned a factor of N.

A cumulant $\langle\!\langle \hat{A}_{\mathrm{L}}(\boldsymbol{k})\, \hat{A}_{\mathrm{L}}(\boldsymbol{k}') \rangle\!\rangle_{\mathrm{L}}$ with respect to the local equilibrium distribution is defined as

$$\begin{aligned}
\langle\!\langle \hat{A}_{\mathrm{L}}(\boldsymbol{k})\, \hat{A}_{\mathrm{L}}(\boldsymbol{k}') \rangle\!\rangle_{\mathrm{L}} &= \left[\frac{\delta^2}{\delta\lambda_{\boldsymbol{k}}\, \delta\lambda_{\boldsymbol{k}'}} \ln\Big\langle \exp\Big(\sum_k A(k) \cdot \lambda_k \Big) \Big\rangle_{\mathrm{L}} \right]_{\lambda=0} \\
&= \left[\frac{\delta^2}{\delta\lambda_k\, \delta\lambda_{k'}} \ln\Big\langle \exp\Big(\sum_k A(k) \cdot \lambda_k + \frac{1}{V} \sum_{\boldsymbol{q}} A(\boldsymbol{q})^* \cdot \boldsymbol{\phi}^{(1)}(\boldsymbol{q}) \Big) \Big\rangle \right]_{\lambda=0} \\
&= \sum_{l,m}{}' \sum_{\boldsymbol{q}^l} \frac{C_{lm}}{V^l} \Big(\frac{\delta^2}{\delta\lambda_k\, \delta\lambda_{\boldsymbol{k}'}} \langle\!\langle \hat{A}_{\mathrm{L}}(\boldsymbol{k}_1) \cdots \hat{A}_{\mathrm{L}}(\boldsymbol{k}_m)\, \hat{A}_{\mathrm{L}}(\boldsymbol{q}_1)^* \cdots \hat{A}_{\mathrm{L}}(\boldsymbol{q}_l)^* \rangle\!\rangle \\
&\quad \times \boldsymbol{\phi}^{(1)}(\boldsymbol{q}_1) \cdots \boldsymbol{\phi}^{(1)}(\boldsymbol{q}_l)\, \lambda_{\boldsymbol{k}_1} \cdots \lambda_{\boldsymbol{k}_m} \Big)_{\lambda=0}\ ,
\end{aligned} \tag{3.15}$$

where the prime on the sum over l and m indicates that $l+m>0$, $\langle\!\langle\ \rangle\!\rangle$ denotes a cumulant with respect to the equilibrium ensemble, and C_{lm} is some counting factor. After taking the functional derivatives and setting λ to zero, we obtain

$$\langle\!\langle \hat{A}_{\mathrm{L}}(\boldsymbol{k})\hat{A}_{\mathrm{L}}(\boldsymbol{k}')\rangle\!\rangle_{\mathrm{L}} = \sum_{l} \frac{C_{l2}}{V^{l}} \sum_{\boldsymbol{q}^{l}} \langle\!\langle \hat{A}_{\mathrm{L}}(\boldsymbol{k})\, \hat{A}_{\mathrm{L}}(\boldsymbol{k}')\, \hat{A}_{\mathrm{L}}(\boldsymbol{q}_1)^{*} \cdots \hat{A}_{\mathrm{L}}(\boldsymbol{q}_l)^{*}\rangle\!\rangle \times \boldsymbol{\phi}^{(1)}(\boldsymbol{q}_1)\cdots\boldsymbol{\phi}^{(1)}(\boldsymbol{q}_l)\,. \tag{3.16}$$

Now $\langle\!\langle \hat{A}_{\mathrm{L}}(\boldsymbol{k})\, \hat{A}_{\mathrm{L}}(\boldsymbol{k}')\, \hat{A}_{\mathrm{L}}(\boldsymbol{q}_1)^{*} \cdots \hat{A}_{\mathrm{L}}(\boldsymbol{q}_l)^{*}\rangle\!\rangle \sim O(N)$ by assignment, and translational invariance requires that $\boldsymbol{q}_l = \boldsymbol{k} + \boldsymbol{k}' - \boldsymbol{q}_1 - \cdots - \boldsymbol{q}_{l-1}$, so we find that each term in the expansion of $\langle\!\langle \hat{A}_{\mathrm{L}}(\boldsymbol{k})\hat{A}_{\mathrm{L}}\boldsymbol{k}'\rangle\!\rangle_{\mathrm{L}}$ is of order N times a coefficient of order

$$\begin{aligned}&\sum_{\boldsymbol{q}_1,\ldots,\boldsymbol{q}_{l-1}} \int \frac{\mathrm{d}\boldsymbol{R}_1\cdots\mathrm{d}\boldsymbol{R}_l}{V^{l}}\, \boldsymbol{\phi}^{(1)}(\boldsymbol{R}_1)\cdots\boldsymbol{\phi}^{(1)}(\boldsymbol{R}_l)\\ &\quad\times \exp[\mathrm{i}\boldsymbol{q}_1\cdot(\boldsymbol{R}_1-\boldsymbol{R}_l)]\cdots\exp[\mathrm{i}\boldsymbol{q}_{l-1}\cdot(\boldsymbol{R}_{l-1}-\boldsymbol{R}_l)]\exp[-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{R}_l]\\ &\quad= \frac{1}{V}\int \mathrm{d}\boldsymbol{R}\, \boldsymbol{\phi}^{(1)l}(\boldsymbol{R})\exp[-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{R}] \sim \mathrm{O}(1)\,,\end{aligned} \tag{3.17}$$

and hence we conclude that

$$\langle\!\langle \hat{A}_{\mathrm{L}}(\boldsymbol{k})\, \hat{A}_{\mathrm{L}}(\boldsymbol{k}')\rangle\!\rangle_{\mathrm{L}} \sim \mathrm{O}(N)\,. \tag{3.18}$$

Following similar lines of argument [12], it can be shown that in fact all cumulants with respect to the local equilibrium distribution are of the same order as the cumulants with respect to the equilibrium distribution, and therefore we will designate each local equilibrium cumulant as order N.

Furthermore, since

$$\begin{aligned}\int \mathrm{d}\boldsymbol{r}'\, A(\boldsymbol{r}')\cdot\boldsymbol{\phi}^{(1)}(\boldsymbol{r}',t) &= \int \mathrm{d}\boldsymbol{r}'\, A(\boldsymbol{r}')\cdot\boldsymbol{\phi}^{(1)}(\boldsymbol{r},t)\\ &\quad+ \int \mathrm{d}\boldsymbol{r}'\, A(\boldsymbol{r}')\cdot[\boldsymbol{\phi}^{(1)}(\boldsymbol{r}',t)-\boldsymbol{\phi}^{(1)}(\boldsymbol{r},t)]\,,\end{aligned} \tag{3.19}$$

we may write the local equilibrium average of a multilinear density as

$$\begin{aligned}\langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_2)\rangle_{\mathrm{L}} &= \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_2)\rangle_{\mathrm{H}}(\boldsymbol{R},t)\\ &\quad+ \int \mathrm{d}\boldsymbol{r}'\, \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_2)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}')\rangle_{\mathrm{L}}\cdot[\boldsymbol{\phi}^{(1)}(\boldsymbol{r}',t)-\boldsymbol{\phi}^{(1)}(\boldsymbol{R},t)] + \cdots\,,\end{aligned} \tag{3.20}$$

where $\boldsymbol{R} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)$,

$$\langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_2)\rangle_{\mathrm{H}}(\boldsymbol{R}, t) \equiv \frac{\langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_2) \exp[A \cdot \boldsymbol{\phi}^{(1)}(\boldsymbol{R}, t)]\rangle}{\langle \exp[A \cdot \boldsymbol{\phi}^{(1)}(\boldsymbol{R}, t)]\rangle}\,, \tag{3.21}$$

and $A \equiv \int \mathrm{d}\boldsymbol{r}\, A(\boldsymbol{r})$. The correlation function in the second term on the right hand side of eq. (3.20) is zero for $|\boldsymbol{r}' - \boldsymbol{R}| > a$, where a is the local equilibrium correlation length, so the overall term is of order $(a/L)^2 \sim (ak)^2 \ll 1$, where L is the length scale on which $\boldsymbol{\phi}^{(1)}(\boldsymbol{R}, t)$ varies. Thus, to a very good approximation,

$$\begin{aligned}\langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_2)\rangle_{\mathrm{L}} &= \langle \hat{A}_{\mathrm{L}}(\boldsymbol{R} + \tfrac{1}{2}\boldsymbol{r}_{12})\, \hat{A}_{\mathrm{L}}(\boldsymbol{R} - \tfrac{1}{2}\boldsymbol{r}_{12})\rangle_{\mathrm{H}}(\boldsymbol{R}, t) + \mathrm{O}(a/L)^2 \\ &= \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_{12})\, \hat{A}_{\mathrm{L}}(0)\rangle_{\mathrm{H}}(\boldsymbol{R}, t)\,,\end{aligned} \tag{3.22}$$

where $\langle \hat{A}_{\mathrm{L}}(\boldsymbol{R} + \frac{1}{2}\boldsymbol{r}_{12})\, \hat{A}_{\mathrm{L}}(\boldsymbol{R} - \frac{1}{2}\boldsymbol{r}_{12})\rangle_{\mathrm{H}}(\boldsymbol{R}, t)$ indicates the average of the bilinear product $\hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_2)$ over the homogeneous distribution function in which the thermodynamic forces $\boldsymbol{\phi}^{(1)}(\boldsymbol{r}', t)$ are evaluated at the center of mass coordinate of the multilinear densities $\boldsymbol{R}$. Note that in Fourier space this implies

$$\begin{aligned}&\langle \hat{A}_{\mathrm{L}}(\boldsymbol{q} + \tfrac{1}{2}\boldsymbol{k})\, \hat{A}_{\mathrm{L}}(\boldsymbol{q} - \tfrac{1}{2}\boldsymbol{k})^*\rangle_{\mathrm{L}} \\ &\quad = \frac{\langle \hat{A}_{\mathrm{L}}(\boldsymbol{q})\, \hat{A}_{\mathrm{L}}(\boldsymbol{q})^*\rangle_{\mathrm{H}}(\boldsymbol{k}, t)}{V} + \mathrm{O}((ka)^2)\,,\end{aligned} \tag{3.23}$$

where $\boldsymbol{q}$ is the wavevector corresponding to the relative spatial variable $\boldsymbol{r}_{12}$ and $\boldsymbol{k}$ is the wavevector corresponding to the center of mass coordinate $\boldsymbol{R}$. Note that unlike the equilibrium correlation function which is obtained from eq. (3.23) by setting $\boldsymbol{\phi}^{(1)}(t) = 0$, the local equilibrium average depends on k. By expanding (3.23) in a Taylor series around the equilibrium average, we see that

$$\begin{aligned}&\langle \hat{A}_{\mathrm{L}}(\boldsymbol{q} + \tfrac{1}{2}\boldsymbol{k})\, \hat{A}_{\mathrm{L}}(\boldsymbol{q} - \tfrac{1}{2}\boldsymbol{k})^*\rangle_{\mathrm{L}} = \langle \hat{A}(\boldsymbol{q})\, \hat{A}(\boldsymbol{q})^*\rangle\, \delta_{k,0} \\ &\quad + \int \mathrm{d}\boldsymbol{R}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}} \langle \hat{A}(\boldsymbol{q})\, \hat{A}(\boldsymbol{q})^* \hat{A}\rangle \cdot \frac{\boldsymbol{\phi}^{(1)}(\boldsymbol{R}, t)}{V} + \cdots\,,\end{aligned} \tag{3.24}$$

which demonstrates that the k dependence arises through terms like

$$\int \mathrm{d}\boldsymbol{R}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}} \boldsymbol{\phi}^{(1)}(\boldsymbol{R}, t)\,.$$

As in the case of equilibrium correlation functions, the local equilibrium averages may be expanded in terms of its cumulants. For example, the bilinear–bilinear correlation function $\langle Q_2 Q_2 \rangle_{\mathrm{L}}$ may be written as

$$\begin{aligned}\langle Q_2(\boldsymbol{r}_1, \boldsymbol{r}_2)\, Q_2(\boldsymbol{r}_3, \boldsymbol{r}_4)\rangle_{\mathrm{L}} &= \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_3)\rangle_{\mathrm{L}} \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_2)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_4)\rangle_{\mathrm{L}} \\ &\quad + \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_4)\rangle_{\mathrm{L}} \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_2)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_3)\rangle_{\mathrm{L}} + \text{additional terms}\,. \qquad (3.25)\end{aligned}$$

In Fourier space, the first two terms of eq. (3.25) are of order N^2 whereas the additional correction terms are of order N. Due to the homogeneous nature of the local equilibrium averages in hydrodynamic systems, we expect that the unfactored $\langle Q_2 Q_2 \rangle_{\mathrm{L}}$ depends on the center of mass coordinate $\boldsymbol{R} \equiv \frac{1}{4}(\boldsymbol{r}_1 + \cdots + \boldsymbol{r}_4)$ only through the thermodynamic forces $\boldsymbol{\phi}^{(1)}(\boldsymbol{R}, t)$ which appear in the homogeneous distribution function. Similarly, the linear–linear correlation function $\langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_3)\rangle_{\mathrm{L}}$ depends on $\boldsymbol{R}_1 \equiv \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_3)$ only through the thermodynamic forces $\boldsymbol{\phi}^{(1)}(\boldsymbol{R}_1, t)$.

We now define the $\sigma_l(t)$ generalized local equilibrium distribution function to be

$$\sigma_l(t) = \frac{W_{\mathrm{e}} \exp(\Sigma_{\alpha=0}^{l}\, Q_\alpha * \boldsymbol{\phi}^{(\alpha)})}{\langle \exp(\Sigma_{\alpha=0}^{l}\, Q_\alpha * \boldsymbol{\phi}^{(\alpha)})\rangle}\,, \qquad (3.26)$$

and define $\langle B \rangle_l \equiv \int \mathrm{d}X_t\, B(X_t)\, \sigma_l(t)$. Note that the local equilibrium distribution is just $\sigma_l(t)$ with $l = 1$. It was established in eq. (3.12) that $\langle Q_l \rangle_t = \overline{Q}_l(t)$, but since $\langle Q_l \rangle_t$ involves all orders α of multilinear densities Q_α and generalized forces $\boldsymbol{\phi}^{(\alpha)}(t)$, it is clear that the equation of motion for the nonequilibrium averages of the Q are coupled in an infinite hierarchy of equations. The N-ordering scheme may be applied to $\langle Q_l \rangle_t$ to simplify this hierarchy and make the formalism tractable.

We have already established that local equilibrium cumulants are of order N,

$$\langle A(\boldsymbol{k})\rangle_{\mathrm{L}} = \langle\!\langle A(\boldsymbol{k})\rangle\!\rangle_{\mathrm{L}} \sim \mathrm{O}(N)\,. \qquad (3.27)$$

We can expand the average of $A(\boldsymbol{k})$ over the $l = 2$ local equilibrium distribution in terms of a series of local equilibrium averages to obtain

$$\begin{aligned}\langle A(\boldsymbol{k})\rangle_2 &= \langle A(\boldsymbol{k})\rangle_{\mathrm{L}} + \langle A(\boldsymbol{k}) Q_2\rangle_{\mathrm{L}} * \boldsymbol{\phi}^{(2)} + \tfrac{1}{2}\langle \hat{A}_{\mathrm{L}}(\boldsymbol{k})\, Q_2 Q_2\rangle_{\mathrm{L}} \,{}^{*}_{*}\, \boldsymbol{\phi}^{(2)}\boldsymbol{\phi}^{(2)} \\ &\quad + \mathrm{O}(\boldsymbol{\phi}^{(2)3})\,.\end{aligned}$$

The term linearly proportional to $\boldsymbol{\phi}^{(2)}$ vanishes by definition of Q_2, and the next term is of leading N order,

$$
\begin{aligned}
&\frac{1}{V^4}\sum_{j,j'}\sum_{l,l'}\langle \hat{A}_{\mathrm{L}}(\boldsymbol{k})\, Q_2(j,j')^*\, Q_2(l,l')^*\rangle_{\mathrm{L}} : \boldsymbol{\phi}^{(2)}(j,j')\,\boldsymbol{\phi}^{(2)}(l,l') \\
&\quad\sim \frac{1}{V^4}\sum_{j,l}\langle \hat{A}_{\mathrm{L}}(k)\,\hat{A}_{\mathrm{L}}(j)^*\,\hat{A}_{\mathrm{L}}(k-j)^*\rangle_{\mathrm{L}}\langle \hat{A}_{\mathrm{L}}(l)\,\hat{A}_{\mathrm{L}}(l)^*\rangle_{\mathrm{L}} \\
&\qquad\times \boldsymbol{\phi}^{(2)}(j,-l)\,\boldsymbol{\phi}^{(2)}(l,k-j) \\
&\quad\sim \frac{N^2}{V^4}\sum_{j,l}\boldsymbol{\phi}^{(2)}(j,-l)\,\boldsymbol{\phi}^{(2)}(l,k-j) \\
&\quad= \frac{N^2}{V^4}\sum_{j,l}\int \mathrm{d}\boldsymbol{r}_1\ldots\mathrm{d}\boldsymbol{r}_4\,\boldsymbol{\phi}^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2)\,\boldsymbol{\phi}^{(2)}(\boldsymbol{r}_3,\boldsymbol{r}_4) \\
&\qquad\times \exp[\mathrm{i}(j\cdot\boldsymbol{r}_1 - l\cdot\boldsymbol{r}_2 + l\cdot\boldsymbol{r}_3 + (k-j)\cdot\boldsymbol{r}_4)]\,.
\end{aligned}
\tag{3.28}
$$

Now by hypothesis $\boldsymbol{\phi}^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2)=\boldsymbol{\psi}^{(2)}(\boldsymbol{R},\boldsymbol{r}_{12})$, where $\boldsymbol{\psi}^{(2)}(\boldsymbol{R},\boldsymbol{r}_{12})$ is a intermediate-ranged function of $\boldsymbol{r}_{12}$. Changing to center of mass and relative variables and summing over $j-l$ gives

$$
\begin{aligned}
&\frac{N^2}{V^3}\sum_{j+1}^{K_\mathrm{c}}\int \mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{R}'\,\mathrm{d}\boldsymbol{r}_{12}\,\mathrm{d}\boldsymbol{r}_{34}\,\delta(\boldsymbol{R}-\boldsymbol{R}')\exp[\mathrm{i}\boldsymbol{k}\cdot(\tfrac{1}{2}\boldsymbol{r}_{34}+\boldsymbol{R})] \\
&\qquad\times\exp[\mathrm{i}(\tfrac{1}{2}\boldsymbol{r}_{12}+\boldsymbol{r}_{34})\cdot(j+l)]\,\boldsymbol{\psi}^{(2)}(\boldsymbol{R},\boldsymbol{r}_{12})\,\boldsymbol{\psi}^{(2)}(\boldsymbol{R}',\boldsymbol{r}_{34})\,.
\end{aligned}
\tag{3.29}
$$

Now, $\boldsymbol{r}_{12}$ and $\boldsymbol{r}_{34}$ are restricted such that $|\boldsymbol{r}_{12}|$, $|\boldsymbol{r}_{34}|<\xi$ by the intermediate-ranged nature of $\boldsymbol{\psi}^{(2)}$, and since $k\xi \ll 1$ for all $k\leqslant K_\mathrm{c}$, we see that

$$
\exp(\tfrac{1}{2}\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_{34}) \approx \exp[\tfrac{1}{2}\mathrm{i}(j+l)\cdot(\boldsymbol{r}_{12}+\boldsymbol{r}_{34})]\approx 1\,,
\tag{3.30}
$$

and hence eq. (3.29) is approximately of N order,

$$
\begin{aligned}
&\frac{N^2}{V^2}\int_0^{K_c}\mathrm{d}\boldsymbol{q}\int \mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{r}_{12}\,\mathrm{d}\boldsymbol{r}_{34}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}}\boldsymbol{\psi}^{(2)}(\boldsymbol{R},\boldsymbol{r}_{12})\,\boldsymbol{\psi}^{(2)}(\boldsymbol{R},\boldsymbol{r}_{34}) \\
&\quad\approx \frac{N^2}{V^2}V(K_\mathrm{c}\xi)^3\xi^3 \equiv N\left(\frac{M}{N}\right)\,.
\end{aligned}
\tag{3.31}
$$

To obtain the N order of (3.31), we have used the facts that

$$
\int \mathrm{d}\boldsymbol{R}\,\boldsymbol{\psi}^{(2)}(\boldsymbol{R},\boldsymbol{r}_{12})\,\boldsymbol{\psi}^{(2)}(\boldsymbol{R},\boldsymbol{r}_{34})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}} \sim \mathrm{O}(V)
\tag{3.32}
$$

and

$$\int \mathrm{d}\boldsymbol{r}_{12}\, \boldsymbol{\psi}^{(2)}(\boldsymbol{R}, \boldsymbol{r}_{12}) \sim \xi^3 \sim \mathrm{O}(1)\,. \tag{3.33}$$

Since $M/N \equiv (K_c \xi)^3$ is small away from critical points, this term is negligible compared to the O(N) term, $\langle A(\boldsymbol{k})\rangle_{\mathrm{L}}$.

Strictly speaking, unlike the factorization of equilibrium correlation functions, the factorization of local equilibrium correlation functions in wavevector space is not accompanied by a delta function between wavevectors since averages over the homogeneous distribution function depend on the center of mass coordinate of the dynamical variables. This dependence on the center of mass coordinate prevents the selection of different N orders of a local equilibrium correlation function by wavevector manipulation since each cumulant in the N order expansion of the correlation function is no longer accompanied by a wavevector equality. However, it is shown in the appendix that the additional sums that exist for the local equilibrium factorizations do not change the overall N order. Henceforth, for simplicity of analysis, when calculating N orders we will assume that factoring a local equilibrium correlation function is accompanied by a delta function in wavevector space.

Analysis of the higher order terms in $\boldsymbol{\phi}^{(2)}$ proceeds similarly and reveals that all the higher order terms in $\boldsymbol{\phi}^{(2)}$ are also of leading order N (M/N), so we may conclude that

$$\langle A(\boldsymbol{k})\rangle_2 = \langle A(\boldsymbol{k})\rangle_{\mathrm{L}}\left[1 + \mathrm{O}\left(\frac{M}{N}\right)\right]. \tag{3.34}$$

In general, it is found that

$$\langle A(\boldsymbol{k})\rangle_t = \langle A(\boldsymbol{k})\rangle_{\mathrm{L}} + \text{corrections}\,, \tag{3.35}$$

where the correction terms are of the general form

$$\langle \hat{A}_{\mathrm{L}}(\boldsymbol{k})\,(\overbrace{Q_2 \cdots Q_2}^{l_2}\,\overbrace{Q_3 \cdots Q_3}^{l_3} \cdots)\rangle_{\mathrm{L}}\, \frac{1}{V^m} \sum_{\boldsymbol{q}^m} \boldsymbol{\phi}^{(2)l_2} \boldsymbol{\phi}^{(3)l_3} \cdots\,, \tag{3.36}$$

in which there are a total number of $n = l_2 + l_3 + \cdots$ Q_i's with $i \geqslant 2$ and $m = 2l_2 + 3l_3 + \cdots$ total number of wavevector sums. Thus there are n center of mass wavevectors and $m - n$ relative wavevectors. After the correlation function is factored to obtain the leading N order term ($\mathrm{O}(N^{m/2}$) if m is even or O ($N^{m-1/2}$) if m is odd), there are only $\frac{1}{2}m$ sums left over the $\boldsymbol{\phi}^{(n)}$'s since each cumulant factor has an approximate delta function in wavevector space associated with it due to translational invariance. Now $n-1$ sums may be carried out over the center of mass wavevectors to give $n-1$ delta functions among the center of mass coordinates, and eq. (3.36) then is of leading order (assuming m is even),

$$\frac{N^{m/2}}{V^m} V^{(n-1)} \sum_{\boldsymbol{q}^{m/2+1-n}} \int \mathrm{d}\boldsymbol{R} \int \mathrm{d}\boldsymbol{r}_{12} \ldots \boldsymbol{\psi}^{(2)}(\boldsymbol{R}, \boldsymbol{r}_{12})^{l_2} \boldsymbol{\psi}^{(3)}(\boldsymbol{R}, \boldsymbol{r}_{34}, \boldsymbol{r}_{35})^{l_3} \cdots , \tag{3.37}$$

which is of leading order,

$$\frac{N^{m/2}}{V^{m/2-1}} \left(\frac{M}{N}\right)^{m/2+1-n} (\xi^3)^{m/2-1} = N\left(\frac{M}{N}\right)^{(m-2n)/2+1} .$$

Since $m-2n \geqslant l_3 + 2l_4 + \cdots \geqslant 0$, we finally obtain the fundamental result

$$\langle A(\boldsymbol{k})\rangle_t = \langle A(\boldsymbol{k})\rangle_{\mathrm{L}} \left[1 + \mathrm{O}\left(\frac{M}{N}\right)\right] . \tag{3.38}$$

This analysis can be generalized to establish the extremely useful fact that for all $l \geqslant 1$,

$$\langle Q_l\rangle_t = \langle Q_l\rangle_l \left[1 + \mathrm{O}\left(\frac{M}{N}\right)\right] , \tag{3.39}$$

by demonstrating that

$$\begin{aligned} &\langle Q_l(\overbrace{Q_{l+1}\cdots Q_{l+1}}^{l_1}\overbrace{Q_{l+2}\cdots Q_{l+2}}^{l_2}\cdots)\rangle_l \frac{1}{V^m}\sum_{\boldsymbol{q}^m} \boldsymbol{\phi}^{(l+1)l_1}\boldsymbol{\phi}^{(l+2)l_2}\ldots \\ &\quad \leq N\left(\frac{M}{N}\right)^{n(l_1+l_2+\cdots-1)+1} \leq N\left(\frac{M}{N}\right) , \end{aligned} \tag{3.40}$$

since at least one of the $l_i \geqslant 1$. Note that eq. (3.39) implies that to leading order in M/N,

$$\frac{\partial \overline{Q}_l(t)}{\partial t} = \frac{\partial \langle Q_l\rangle_l}{\partial t} , \tag{3.41}$$

so that the equation of motion for $\overline{Q}_l(t)$ involves only the multilinear densities Q_α and generalized thermodynamic forces $\boldsymbol{\phi}^{(\alpha)}(t)$ up to $\alpha = l$. Thus the generalized forces $\boldsymbol{\phi}^{(\alpha)}(t)$ can be solved for systematically order by order which thereby permits the equal-time moments to be calculated.

The nonequilibrium averages $\overline{Q_l}(t)$ evolve to leading order in M/N according to the equation

$$\frac{\partial \overline{Q_l}(t)}{\partial t} = \frac{\partial \langle Q_l \rangle_l}{\partial t} = \int \mathrm{d}X_{\mathrm{t}} \left(\frac{\partial \sigma_l}{\partial t} Q_l \right) + \left\langle \frac{\partial Q_l}{\partial t} \right\rangle_l , \tag{3.42}$$

where, for $l \geqslant 1$,

$$\frac{\partial Q_l}{\partial t} = \frac{\delta Q_l}{\delta \boldsymbol{\phi}^{(1)}(t)} \cdot \dot{\boldsymbol{\phi}}^{(1)}(t) .$$

The functional derivative $\delta Q_l / \delta \boldsymbol{\phi}^{(1)}$ can be written as a sum of Q_m with $m = 0, 1, \ldots, l-1$, so that

$$\tilde{\mathscr{P}}_{l-1}(t) \frac{\delta Q_l}{\delta \boldsymbol{\phi}^{(1)}(t)} = \frac{\delta Q_l}{\delta \boldsymbol{\phi}^{(1)}(t)} , \tag{3.43}$$

where

$$\tilde{\mathscr{P}}_{l-1}(t)\, B \equiv \sum_{m=0}^{l-1} \langle B Q_m \rangle_{\mathrm{L}} * K^{(\mathrm{L})-1}_{m \hat{m}} * Q_{\hat{m}} . \tag{3.44}$$

Since $\langle Q_l Q_m \rangle_{\mathrm{L}} = 0$ for $l > m$ by construction,

$$\left\langle \frac{\delta Q_l}{\delta \boldsymbol{\phi}^{(1)}(t)} Q_m + Q_l \frac{\delta Q_m}{\delta \boldsymbol{\phi}^{(1)}(t)} + Q_l Q_m \hat{A}_{\mathrm{L}} \right\rangle_{\mathrm{L}} = 0 . \tag{3.45}$$

However, $\delta Q_m / \delta \boldsymbol{\phi}^{(1)}(t)$ may be written as a sum of Q_r's with $r = 0, 1, \ldots, m-1$, and since $\langle Q_l Q_r \rangle_{\mathrm{L}} = 0$, we conclude that to leading N order and $l \geqslant 1$

$$\frac{\delta Q_l}{\delta \boldsymbol{\phi}^{(1)}(t)} = \left\langle \frac{\delta Q_l}{\delta \boldsymbol{\phi}^{(1)}(t)} Q_{\hat{m}} \right\rangle_{\mathrm{L}} * K^{(\mathrm{L})-1}_{\hat{m} m} * Q_m , \tag{3.46}$$

where $|\hat{m}| = |m| = |l| - 1$. Since

$$\begin{aligned} \left\langle \frac{\delta Q_l}{\delta \boldsymbol{\phi}^{(1)}(t)} Q_m \right\rangle_{\mathrm{L}} &= -\langle Q_l Q_m \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} \\ &= \overbrace{\langle \hat{A}_{\mathrm{L}} \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} \cdots \langle \hat{A}_{\mathrm{L}} \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}}}^{l \text{ factors}} + \mathrm{O}(N^{l-1}) , \end{aligned} \tag{3.47}$$

we find that to leading N order

$$\begin{aligned}
&\frac{\partial Q_0}{\partial t} = 0\,,\\
&\frac{\partial Q_1}{\partial t} = -\langle \hat{A}_{\mathrm{L}} \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} \cdot \dot{\boldsymbol{\phi}}^{(1)}(t)\,,\\
&\frac{\partial Q_2}{\partial t} = -\hat{A}_{\mathrm{L}} \langle \hat{A}_{\mathrm{L}} \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} \cdot \dot{\boldsymbol{\phi}}^{(1)}(t)\,,
\end{aligned} \tag{3.48}$$

and in general for $l \geqslant 3$

$$\frac{\partial Q_l}{\partial t} = -Q_{l-1} \langle \hat{A}_{\mathrm{L}} \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} \cdot \dot{\boldsymbol{\phi}}^{(1)}(t)\,. \tag{3.49}$$

Similarly, we obtain for $\partial \sigma_l(t)/\partial t$

$$\begin{aligned}
\frac{\partial \sigma_1(t)}{\partial t} &= \sigma_1(t) \hat{A}_{\mathrm{L}} \cdot \dot{\boldsymbol{\phi}}^{(1)}(t)\,,\\
\frac{\partial \sigma_2(t)}{\partial t} &= \sigma_2(t)[\hat{A}_{\mathrm{L}} * \dot{\boldsymbol{\phi}}^{(1)}(t) + (Q_2 - \langle Q_2 \rangle_2) * \dot{\boldsymbol{\phi}}^{(2)}(t)\\
&\quad - \dot{\boldsymbol{\phi}}^{(1)}(t) * \langle \hat{A}_{\mathrm{L}} \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} \hat{A}_{\mathrm{L}} * \boldsymbol{\phi}^{(2)}(t)]\left[1 + \mathrm{O}\left(\frac{M}{N}\right)\right],
\end{aligned} \tag{3.50}$$

and for $l \geqslant 3$

$$\begin{aligned}
\frac{\partial \sigma_l(t)}{\partial t} &= \sigma_l(t) \left(\sum_{n=1}^{l} (Q_n - \langle Q_n \rangle_n) * \dot{\boldsymbol{\phi}}^{(n)}(t) \right.\\
&\quad - \sum_{n=3}^{l} \dot{\boldsymbol{\phi}}^{(1)}(t) * \langle \hat{A}_{\mathrm{L}} \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} (Q_{n-1} - \langle Q_{n-1} \rangle_{n-1}) * \boldsymbol{\phi}^{(n)}(t)\\
&\quad \left. - \dot{\boldsymbol{\phi}}^{(1)}(t) * \langle \hat{A}_{\mathrm{L}} \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} \hat{A}_{\mathrm{L}} * \boldsymbol{\phi}^{(2)}(t) \right) \left[1 + \mathrm{O}\left(\frac{M}{N}\right)\right].
\end{aligned} \tag{3.51}$$

Combining eqs. (3.48), (3.49), (3.50) and (3.51), we finally get

$$\begin{aligned}
\frac{\partial \overline{Q_1}(t)}{\partial t} &= 0\,,\\
\frac{\partial \bar{A}(t)}{\partial t} &= \langle \hat{A}_{\mathrm{L}} \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} * \dot{\boldsymbol{\phi}}^{(1)}(t)\,,\\
\frac{\partial \overline{Q_2}(t)}{\partial t} &= \langle Q_2 \hat{A}_{\mathrm{L}} \rangle_2 * \dot{\boldsymbol{\phi}}^{(1)}(t) + \langle Q_2 (Q_2 - \langle Q_2 \rangle_2) \rangle_2 * \dot{\boldsymbol{\phi}}^{(2)}(t)\\
&\quad - \dot{\boldsymbol{\phi}}^{(1)}(t) * \langle \hat{A}_{\mathrm{L}} \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} \langle Q_2 \hat{A}_{\mathrm{L}} \rangle_2 * \boldsymbol{\phi}^{(2)}(t) + \mathrm{O}\left(N \left(\frac{M}{N}\right)\right),
\end{aligned} \tag{3.52}$$

and for $l \geqslant 3$

$$\begin{aligned}\frac{\partial \overline{Q_l}(t)}{\partial t} = & \sum_{n=1}^{l} \langle Q_l(Q_n - \langle Q_n \rangle_n)\rangle_l * \dot{\boldsymbol{\phi}}^{(n)}(t) \\ & - \sum_{n=3}^{l} \dot{\boldsymbol{\phi}}^{(1)}(t) * \langle \hat{A}_{\mathrm{L}} \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} \langle (Q_{n-1} - \langle Q_{n-1} \rangle_{n-1}) Q_l \rangle_l * \boldsymbol{\phi}^{(n)}(t) \\ & - \dot{\boldsymbol{\phi}}^{(1)}(t) * \langle \hat{A}_{\mathrm{L}} \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} \langle Q_l \hat{A}_{\mathrm{L}} \rangle_l * \boldsymbol{\phi}^{(2)}(t) \\ & - \langle Q_{l-1} \rangle_l \langle \hat{A}_{\mathrm{L}} \hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} * \dot{\boldsymbol{\phi}}^{(1)}(t) + \mathrm{O}\Big(N\Big(\frac{M}{N}\Big)\Big) . \end{aligned} \tag{3.53}$$

Eqs. (3.52) and (3.53) are the fundamental equations which determine equal-time correlations in nonequilibrium systems.

We now define the projection operator $\mathscr{P}^{\dagger}(t)$ by

$$\mathscr{P}^{\dagger}(t)\, G(\boldsymbol{r}) \equiv \sum_{n,\hat{n}=0}^{\infty} \Big(\int \mathrm{d}X_{\mathrm{t}}\, G(\boldsymbol{r})\, Q_n \Big) * K_{n\hat{n}}^{(t)-1} * Q_{\hat{n}} \sigma(t) , \tag{3.54}$$

where $K_{ij}^{(t)} \equiv \langle Q_i Q_j \rangle_t$. It is easily verified that $\mathscr{P}^{\dagger}(t)$ projects the distribution $W(t)$ onto the generalized local equilibrium distribution function $\sigma(t)$,

$$\mathscr{P}^{\dagger}(t)\, W(t) = \sigma(t) ,$$

since for all n

$$\overline{Q_n}(t) = \langle Q_n Q_0 \rangle_t .$$

Furthermore, it is easily established [11] that $\mathscr{P}^{\dagger}(t)$ is the Hermitian adjoint of the projection operator $\mathscr{P}(t)$, where $\mathscr{P}(t)$ is defined by

$$\mathscr{P}(t)\, B \equiv \sum_{n,\hat{n}=0}^{\infty} \langle B Q_n \rangle_t * K_{n\hat{n}}^{(t)-1} * Q_{\hat{n}} , \tag{3.55}$$

so that for arbitrary functions F and G of the phase space X_{t},

$$\int \mathrm{d}X_{\mathrm{t}}\, F[\mathscr{P}(t)\, G] = \int \mathrm{d}X_{\mathrm{t}}\, [\mathscr{P}^{\dagger}(t)\, F]\, G . \tag{3.56}$$

We are interested in obtaining nonlinear equations of motion for the nonequilibrium averages $\bar{Q}(t)$ of our Q basis set. By construction, for all i,

$$\mathscr{P}(t)\, Q_i = Q_i , \tag{3.57}$$

and hence

$$\mathcal{P}^{\dagger}(t)\frac{\partial\sigma(t)}{\partial t}=\frac{\partial\sigma(t)}{\partial t}\,, \tag{3.58}$$

since $\partial\sigma(t)/\partial t$ may be written as $\sigma(t)$ times a linear combination of the Q_i's. Now

$$\mathcal{P}^{\dagger}(t)\frac{\partial W(t)}{\partial t}=\sigma(t)\left[\sum_{n,\hat{n}=0}^{\infty}\left(\int \mathrm{d}X_t\,\frac{\partial W(t)}{\partial t}\,Q_n\right)*K^{(t)-1}_{n\hat{n}}*Q_{\hat{n}}\right], \tag{3.59}$$

but since

$$\int \mathrm{d}X_t\,\frac{\partial W(t)}{\partial t}\,Q_n=\int \mathrm{d}X_t\,\frac{\partial\sigma(t)}{\partial t}\,Q_n\,, \tag{3.60}$$

we can conclude from eq. (3.58) that

$$\mathcal{P}^{\dagger}(t)\frac{\partial W(t)}{\partial t}=\mathcal{P}^{\dagger}(t)\frac{\partial\sigma(t)}{\partial t}=\frac{\partial\sigma(t)}{\partial t}\,. \tag{3.61}$$

If we define $\chi(t)$ to be

$$\chi(t)\equiv[1-\mathcal{P}^{\dagger}(t)]W(t)\equiv\mathcal{Q}^{\dagger}(t)\,W(t)=W(t)-\sigma(t)\,, \tag{3.62}$$

so that $W(t)=\mathcal{P}^{\dagger}(t)W(t)+\mathcal{Q}^{\dagger}(t)W(t)=\sigma(t)+\chi(t)$, then from eq. (3.61) and the fact that

$$\dot{W}(t)=\frac{\partial W(t)}{\partial t}=OW(t)\,,$$

we obtain

$$\frac{\partial\sigma(t)}{\partial t}=\mathcal{P}^{\dagger}(t)\,OW(t)=\mathcal{P}^{\dagger}(t)\,O[\sigma(t)+\chi(t)]\,. \tag{3.63}$$

Now $\dot{W}(t)=OW(t)=\mathcal{P}^{\dagger}(t)\,OW(t)+\mathcal{Q}^{\dagger}(t)\,OW(t)$, so

$$\frac{\partial\chi(t)}{\partial t}=\dot{W}(t)-\frac{\partial\sigma(t)}{\partial t}=\mathcal{Q}^{\dagger}(t)\,OW(t)=\mathcal{Q}^{\dagger}(t)\,O\sigma(t)+\mathcal{Q}^{\dagger}(t)\,O\chi(t)\,. \tag{3.64}$$

The formal solution of eq. (3.64) is

$$\chi(t) = \left[T_+ \exp\left(\int_0^t \mathrm{d}\sigma\, \mathcal{Q}^\dagger(\sigma)\, O\right)\right]\chi(0) + \int_0^t \mathrm{d}\tau \left[T_+ \exp\left(\int_\tau^t \mathrm{d}\sigma\, \mathcal{Q}^\dagger(\sigma)\, O\right)\right]\mathcal{Q}^\dagger(\tau)\, O\sigma(\tau)\,, \tag{3.65}$$

where T_+ is the time ordering operator. If we insert (3.65) in eq. (3.63), we obtain

$$\frac{\partial\sigma(t)}{\partial t} = \mathcal{P}^\dagger(t)\, O\sigma(t) + \mathcal{P}^\dagger(t)\, O \int_0^t \mathrm{d}\tau\, [T_+ \exp\left(\int_\tau^t \mathrm{d}\sigma\, \mathcal{Q}^\dagger(\sigma)\, O\right)\Big]\mathcal{Q}^\dagger(\tau)\, O\sigma(\tau) + \mathcal{P}^\dagger(t)\, O\left[T_+ \exp\left(\int_0^t \mathrm{d}\sigma\, \mathcal{Q}^\dagger(\sigma)\, O\right)\right]\chi(0)\,. \tag{3.66}$$

Now using the fact that $\overline{Q_l}(t) = \langle Q_l\rangle_t$, we see that

$$\frac{\partial\overline{Q_l}(t)}{\partial t} = \int \mathrm{d}X \left(\frac{\delta Q_l}{\delta\boldsymbol{\phi}^{(1)}(t)} \cdot \dot{\boldsymbol{\phi}}^{(1)}(t)\, \sigma(t) + Q_l \frac{\partial\sigma(t)}{\partial t}\right).$$

Defining $\dot{\overline{Q_l}}(t)$ by

$$\dot{\overline{Q_l}}(t) = \frac{\partial\overline{Q_l}(t)}{\partial t} - \overline{\frac{\delta Q_l}{\delta\boldsymbol{\phi}^{(1)}(t)}} \cdot \dot{\boldsymbol{\phi}}^{(1)}(t)\,, \tag{3.67}$$

we obtain from eq. (3.66)

$$\dot{\overline{Q_l}}(t) = \int \mathrm{d}X_t\, Q_l \mathcal{P}^\dagger(t)\, O\sigma(t) + \chi(0)\ \text{term} + \int \mathrm{d}X_t\, Q_l \mathcal{P}^\dagger(t)\, O \int_0^t \mathrm{d}\tau \left[T_+ \exp\left(\int_\tau^t \mathrm{d}\sigma\, \mathcal{Q}^\dagger(\sigma)\, O\right)\right]\mathcal{Q}^\dagger(\tau)\, O\sigma(\tau)\,. \tag{3.68}$$

It should be noted that since

$$\int \mathrm{d}X\, Q \frac{\partial\sigma(t)}{\partial t} = \int \mathrm{d}X\, Q \frac{\partial W(t)}{\partial t} = \int \mathrm{d}X\, QOW(t)\,,$$

we see that

$$\overline{\dot{Q}_l(t)} = \overline{O^\dagger Q_l(t)}\ .$$

We shall assume that any slowly varying process of the system is an analytic function of the hydrodynamic densities which compose the set A, and hence for any arbitrary dynamical variables $B(\boldsymbol{r}_1)$ and $C(\boldsymbol{r}_2)$, we expect

$$\langle \{\exp\{[1-\mathscr{P}(t)]O^\dagger t\}\,[1-\mathscr{P}(t)]B(\boldsymbol{r}_1)\}C(\boldsymbol{r}_2)\rangle_{\mathrm{L}} \rightarrow 0 \tag{3.69}$$

for times longer than the timescale for nonhydrodynamic processes. We may rewrite the $\chi(0)$ term of eq. (3.68) as

$$\begin{aligned}&\int \mathrm{d}X_{\mathrm{t}} \left\{O\left[T_+ \exp\left(\int_0^t \mathrm{d}\sigma\, \mathscr{Q}^\dagger(\sigma)\, O\right)\right]\chi(0)\right\} Q_n * K_{n\hat{n}}^{(t)-1} * \langle Q_{\hat{n}} Q_l\rangle_t \\ &\quad = \int \mathrm{d}X_{\mathrm{t}}\, \chi(0) \left[T_- \exp\left(\int_0^t \mathrm{d}\sigma\, O^\dagger \mathscr{Q}(\sigma)\right) O^\dagger Q_l\right],\end{aligned} \tag{3.70}$$

which we expect to decay to zero for $t \gg \tau_{\mathrm{m}}$ since all the slowly varying components of $O^\dagger Q_l$ have been removed by the projection operator $1 - \mathscr{P}(\sigma) = \mathscr{Q}(\sigma)$.

Now since $\int \mathrm{d}X_{\mathrm{t}}\, Q_l \mathscr{P}^\dagger(t) B(\boldsymbol{r}) = \int \mathrm{d}X_{\mathrm{t}}\, B(\boldsymbol{r}) Q_l$, for times longer than τ_{m} we may rewrite eq. (3.68) as

$$\begin{aligned}\overline{\dot{Q}_l(t)} &= \int \mathrm{d}X_{\mathrm{t}}\, Q_l [O\sigma(t)] \\ &\quad + \int \mathrm{d}X_{\mathrm{t}}\, Q_l O \int_0^t \mathrm{d}\tau \left[T_+ \exp\left(\int_\tau^t \mathrm{d}\sigma\, \mathscr{Q}^\dagger(\sigma)\, O\right)\right] \mathscr{Q}^\dagger(\tau)\, O\sigma(\tau) \\ &= \int \mathrm{d}X_{\mathrm{t}}\, (O^\dagger Q_l)\sigma(t) + \int \mathrm{d}X_{\mathrm{t}}\, (O^\dagger Q_l) \\ &\quad \times \int_0^t \mathrm{d}\tau \left[T_+ \exp\left(\int_\tau^t \mathrm{d}\sigma\, \mathscr{Q}^\dagger(\sigma)\, O\right)\right] \mathscr{Q}^\dagger(\tau)\, O\sigma(\tau)\ ,\end{aligned} \tag{3.71}$$

where $O^\dagger$ is given by eq. (2.20). Note that $(\Omega_1 + \Omega_1^\dagger)B = \beta/m\ \Sigma_j\, \Sigma_{k\neq j}\, \gamma_{jk} B$ and hence

$$\begin{aligned}O &= -L - \Omega_1^\dagger + (\Omega_1 + \Omega_1^\dagger) + \Omega_2 \\ &= -O_1^\dagger + \sum_j \sum_{k\neq j} \left(\gamma_{jk}\frac{\beta}{m}\right) + \tfrac{1}{2} \sum_j \sum_{k\neq j} \gamma_{jk} \hat{\boldsymbol{r}}_{jk} \hat{\boldsymbol{r}}_{jk} : (\boldsymbol{\nabla}_{p_j} - \boldsymbol{\nabla}_{p_k})(\boldsymbol{\nabla}_{p_j} - \boldsymbol{\nabla}_{p_k})\ .\end{aligned} \tag{3.72}$$

Since $\mathcal{Q}^{\dagger}O\sigma(\tau) = [\mathcal{Q}(\tau)\,\psi(\tau)]\sigma(\tau)$, where $\psi(\tau)$ is defined by

$$O\sigma(\tau) \equiv \psi(\tau)\,\sigma(\tau)\,, \tag{3.73}$$

we may rewrite eq. (3.71) as

$$\begin{aligned}\overline{\dot{Q}_l}(t) &= \langle O^{\dagger}Q_l\rangle_t \\ &\quad + \int_0^t \mathrm{d}\tau \left\langle \left\{ \left[T_- \exp\left(\int_\tau^t \mathrm{d}\sigma\, \mathcal{Q}(\sigma)\, O^{\dagger} \right) \right] \mathcal{Q}(\tau)\, O^{\dagger} Q_l \right\} [\mathcal{Q}(\tau)\,\psi(\tau)] \right\rangle_\tau .\end{aligned} \tag{3.74}$$

It turns out there is a lot of flexibility in the definition of the projection operator $\mathcal{P}(t)$ which appears in eq. (3.74). If we designate the complete basis set of multilinear hydrodynamic densities by C (it can be the basis set Q), we can write the projection operator $\mathcal{P}(t)$ acting on an arbitrary density D as

$$\mathcal{P}(t)\,D = \langle DC\rangle_t * \langle CC\rangle_t^{-1} * C\,. \tag{3.75}$$

If we take the functional derivative of (3.75) with respect to the generalized forces $\boldsymbol{\phi}(t)$ which appear in (3.75) through the distribution function $\sigma(t)$, we obtain

$$\begin{aligned}\left(\frac{\delta\mathcal{P}(t)\,B}{\delta\boldsymbol{\phi}(t)}\right)' &= \langle DCQ\rangle_t * \langle CC\rangle_t^{-1} * C \\ &\quad - \langle DC\rangle_t * \langle CC\rangle_t^{-1} * \langle CCQ\rangle_t * \langle CC\rangle_t^{-1} * C \\ &= \langle (D - \langle DC\rangle_t * \langle CC\rangle_t^{-1} * C)CQ\rangle_t * \langle CC\rangle_t^{-1} * C \\ &= \langle \{[1-\mathcal{P}(t)]D\}CQ\rangle_t * \langle CC\rangle_t^{-1} * C\,,\end{aligned} \tag{3.76}$$

where the prime on the functional derivative denotes the restriction that the derivative with respect to the $\boldsymbol{\phi}(t)$ operates only on the distribution function $\sigma(t)$. However, we may rewrite CQ as a sum of C_i since C is a complete basis set, which implies that

$$\left(\frac{\delta\mathcal{P}(t)\,B}{\delta\boldsymbol{\phi}(t)}\right)' = 0\,, \tag{3.77}$$

since $\langle\{[1-\mathcal{P}(t)]D\}C_i\rangle_t = 0$ by construction. Thus we conclude that the projection $\mathcal{P}(t)\,D$ *does not depend* on the generalized forces which appear through the distribution function, and hence

$$\mathcal{P}(t)\,D = \mathcal{P}_{\mathrm{L}}(t)\,D\,, \tag{3.78}$$

where $\mathcal{P}_L(t)\,D$ represents the projection operator defined with respect to the local equilibrium distribution $\sigma_1(t)$.

Furthermore, if we assume that the set A of hydrodynamic densities contains all the slow variables of the system which vary on a hydrodynamic timescale τ_H which is long compared to the relaxation time τ_m of the other translational variable, we may approximate eq. (3.74) as

$$\dot{\overline{Q}}_l(t) = \langle O^\dagger Q_l \rangle_t + \int_0^\infty \mathrm{d}\tau \, \langle [G_L(\tau)\, O^\dagger Q_l]\, \mathcal{Q}_L(t)\, \psi(t) \rangle_t + \mathrm{O}\left(\frac{\tau_m}{\tau_H}\right) , \tag{3.79}$$

where

$$G_L(\tau) \equiv \exp[\mathcal{Q}_L(t) O^\dagger \tau]\, \mathcal{Q}_L(t) .$$

Eq. (3.79) may be simplified to leading order in M/N so that the equation of motion for $\overline{Q}_l(t)$ involves only the Q_m and $\boldsymbol{\phi}^{(m)}$ up to $m = l$. The Euler term $\langle O^\dagger Q_l \rangle_t$ can be written as $\langle O^\dagger Q_l \rangle_l$ plus correction terms of the form

$$\langle (O^\dagger Q_l) Q_{l+1}^{l_1} Q_{l+2}^{l_2} \cdots \rangle_l \frac{1}{V^m} \sum_{q^m} \boldsymbol{\phi}^{(l+1)l_1} \boldsymbol{\phi}^{(l+2)l_2 \cdots} . \tag{3.80}$$

Now

$$O^\dagger(Q_1 Q_1) = (O^\dagger Q_1) Q_1 + Q_1 (O^\dagger Q_1) + \Omega_2'(Q_1 Q_1) ,$$

where $\Omega_2'(Q_1 Q_1)$ is defined by

$$\Omega_2'(Q_1 Q_1) \equiv \sum_j \sum_{k \neq j} \gamma_{jk} \hat{\boldsymbol{r}}_{jk} \hat{\boldsymbol{r}}_{jk} : [(\boldsymbol{\nabla}_{p_j} - \boldsymbol{\nabla}_{p_k}) Q_1 (\boldsymbol{\nabla}_{p_j} - \boldsymbol{\nabla}_{p_k}) Q_1] .$$

As was shown in eq. (2.26) of the previous section, explicit evaluation of $\Omega_2'(Q_1(\boldsymbol{r}_1)\, Q_1(\boldsymbol{r}_2))$ yields a short-ranged function of r_{12} which vanishes for $|\boldsymbol{r}_{12}| \equiv r_{12} > a$, where a is a small correlation length. Since the spatial arguments of the Ω_2' cannot be separated, terms containing Ω_2' cannot be factored and therefore give terms of lower N order than the other components of $O^\dagger Q$. Thus, to leading N order the nonlinear operators $O^\dagger$ and O behave linearly. Since the subtraction terms in Q_l also give lower order terms, the leading order terms of $O^\dagger Q_l$ comes from $(O^\dagger Q_1) Q_1^{l-1}$, which obeys the same N order rules as Q_l, and hence

$$\langle O^\dagger Q_l \rangle_t = \langle O^\dagger Q_l \rangle_l \left[1 + \mathrm{O}\left(\frac{M}{N}\right) \right] . \tag{3.81}$$

Now the dissipative term of eq. (3.79) is

$$\int_0^\infty \mathrm{d}\tau \, \langle [G_{\mathrm{L}}(\tau)\, O^{\dagger} Q_l] \psi(t) \rangle_t \,,$$

which is bound above N order by

$$\int_0^\infty \mathrm{d}\tau \, \langle [G_{\mathrm{L}}(\tau)\, O^{\dagger} Q_l] \psi(t) \rangle_{\mathrm{L}} \,.$$

Noting that

$$\begin{aligned}
[1 - \mathcal{P}_{\mathrm{L}}(t)](BQ_1) &= BQ_1 - \langle BQ_1 \rangle_{\mathrm{L}} - \sum_{n=1}^{\infty} \langle BQ_1 Q_n \rangle_{\mathrm{L}} * K_{n\hat{n}}^{(\mathrm{L})^{-1}} * Q_{\hat{n}} \\
&= BQ_1 - \langle BQ_1 \rangle_{\mathrm{L}} - \langle BQ_1 Q \rangle_{\mathrm{L}} * K^{(\mathrm{L})^{-1}} * QQ_1 + \mathrm{O}\left(\frac{M}{N}\right) \\
&= \{[1 - \mathcal{P}_{\mathrm{L}}(t)]B\} Q_1 - \langle BQ_1 \rangle_{\mathrm{L}}
\end{aligned} \qquad (3.82)$$

and in general

$$[1 - \mathcal{P}_{\mathrm{L}}(t)](BQ_j) = \{[1 - \mathcal{P}_{\mathrm{L}}(t)]B\} Q_j - \langle BQ_j \rangle_{\mathrm{L}} + \mathrm{O}\left(\frac{M}{N}\right) , \qquad (3.83)$$

which follow from the fact that

$$\begin{aligned}
Q_l &= Q_j Q_{l-j} - \sum_{n=\max\{2j-l,l-2j\}}^{l-1} \langle Q_j Q_{l-j} Q_n \rangle_{\mathrm{L}} * K_{n\hat{n}}^{(\mathrm{L})-1} * Q_{\hat{n}} \\
&= Q_j Q_{l-j} - \text{lower order terms} \,,
\end{aligned} \qquad (3.84)$$

we see that for a general correlation function of densities B and D,

$$\begin{aligned}
&\langle \{[1 - \mathcal{P}_{\mathrm{L}}(t)] BQ_l\} DQ_{l-1} \rangle_{\mathrm{L}} \approx \langle \{[1 - \mathcal{P}_{\mathrm{L}}(t)]B\} \hat{A}_{\mathrm{L}} D \rangle_{\mathrm{L}} \langle Q_{l-1} Q_{l-1} \rangle_{\mathrm{L}} \,, \\
&\langle \{[1 - \mathcal{P}_{\mathrm{L}}(t)] BQ_l\} DQ_l \rangle_{\mathrm{L}} \approx \langle \{[1 - \mathcal{P}_{\mathrm{L}}(t)]B\} D \rangle_{\mathrm{L}} \langle Q_l Q_l \rangle_{\mathrm{L}} \,, \\
&\langle \{[1 - \mathcal{P}_{\mathrm{L}}(t)] BQ_l\} DQ_{l+1} \rangle_{\mathrm{L}} \approx \langle \{[1 - \mathcal{P}_{\mathrm{L}}(t)]B\} D\hat{A}_{\mathrm{L}} \rangle_{\mathrm{L}} \langle Q_l Q_l \rangle_{\mathrm{L}} \,,
\end{aligned} \qquad (3.85)$$

since $\langle \{[1 - \mathcal{P}_{\mathrm{L}}(t)]B\} Q \rangle_{\mathrm{L}} = 0$ by construction. Now

$$\psi(t) = \psi_l(t) + R_l(t) \,, \qquad (3.86)$$

where $O\sigma_l(t) = \psi_l(t)\, \sigma_l(t)$ and $R_l(t)$ is given by

$$R_l(t) = [1 - \mathscr{P}_{\mathrm{L}}(t)] \sum_{r=l+1}^{\infty} \Big(O^* Q_r * \phi^{(r)} + 2 \sum_{s=l+1}^{\infty}{}' \, \Omega_2'(Q_r Q_s)_*^* \boldsymbol{\phi}^{(r)} \boldsymbol{\phi}^{(s)} + \Omega_2'(Q_r Q_r)_*^* \boldsymbol{\phi}^{(r)2} \Big) , \tag{3.87}$$

with $O^* \equiv -L - \Omega_1^\dagger + \Omega_2$ and the prime on the sum over s indicates that $r \neq s$. As in the case of the Euler term, the operators $O^\dagger$ and O are linear in the leading order factorization of the dissipative term, and the N order of the dissipative term with the factor $R_l(t)$ can be evaluated symbolically as

$$\mathrm{O}(N) \times \langle Q_{l-1}(\tau) \, (Q_{l+1}^{l_1} Q_{l+2}^{l_2} \cdots) \rangle_l \frac{1}{V^m} \sum_{q^m} \boldsymbol{\phi}^{(l+1)l_1} \boldsymbol{\phi}^{(l+2)l_2} \cdots \leq N \left(\frac{M}{N} \right) , \tag{3.88}$$

since $G_{\mathrm{L}}(\tau) = [1 - \mathscr{P}_{\mathrm{L}}(t)] G_{\mathrm{L}}(\tau)$ by construction. Thus, to leading order in M/N,

$$\langle [G_{\mathrm{L}}(\tau) \, O^\dagger Q_l] \psi(t) \rangle_t = \langle [G_{\mathrm{L}}(\tau) \, O^\dagger Q_l] \psi_l(t) \rangle_t \left[1 + \mathrm{O} \left(\frac{M}{N} \right) \right] . \tag{3.89}$$

Similarly, it can be shown that

$$\langle [G_{\mathrm{L}}(\tau) \, O^\dagger Q_l] \psi_l(t) \rangle_t = \langle [G_{\mathrm{L}}(\tau) \, O^\dagger Q_l] \psi_l(t) \rangle_l \left[1 + \mathrm{O} \left(\frac{M}{N} \right) \right] , \tag{3.90}$$

by expanding the $\langle \ \rangle_t$ averages in terms of local equilibrium averages and analyzing the N orders of the various terms. Thus, to leading order in M/N, eq. (3.79) may be written as

$$\overline{\dot{Q}_l}(t) = \langle O^\dagger Q_l \rangle_l + \int_0^\infty \mathrm{d}\tau \langle [G_{\mathrm{L}}(\tau) \, O^\dagger Q_l] \psi_l(t) \rangle_l + \mathrm{O} \left(\frac{M}{N} \right) . \tag{3.91}$$

In order to make contact with earlier work [12–15], we rewrite eq. (3.79) in the form

$$\begin{aligned} \overline{\dot{Q}_l}(t) &= \langle (O^\dagger Q_l) Q \rangle_t * K^{(t)-1} * \bar{Q}(t) \\ &\quad + \int_0^\infty \mathrm{d}\tau \, \langle [O^\dagger G_{\mathrm{L}}(\tau) O^\dagger Q_l] Q \rangle_t * K^{(t)-1} * \bar{Q}(t) \\ &= \overline{\mathscr{P}(t) \, O^\dagger Q_l}(t) + \int \mathrm{d}X_{\mathrm{t}} \, W(t) \int_0^\infty \mathrm{d}\tau \, \mathscr{P}(t) \, O^\dagger G_{\mathrm{L}}(\tau) \, O^\dagger Q_l , \end{aligned} \tag{3.92}$$

using the fact that $\bar{Q}(t) = \langle QQ_0 \rangle_t$. Since $\mathscr{P}(t)\, D = \mathscr{P}_{\mathrm{L}}(t)\, D$, we can rewrite this expression as

$$\dot{\overline{Q_l}}(t) = \int \mathrm{d}X_{\mathrm{t}}\, W(0)\, \mathrm{e}^{O^\dagger t} \mathscr{P}_{\mathrm{L}}(t)\, O^\dagger Q_l + \int \mathrm{d}X_{\mathrm{t}}\, W(0)\, \mathrm{e}^{O^\dagger t} \int_0^\infty \mathrm{d}\tau\, \mathscr{P}_{\mathrm{L}}(t)\, O^\dagger G_{\mathrm{L}}(\tau)\, O^\dagger Q_l = \sum_{m=0}^{\infty} \tilde{M}_{lm}(t) * \overline{Q_m(t)}^{\,0}, \tag{3.93}$$

where $Q_l(t) \equiv \mathrm{e}^{O^\dagger t} Q_l$,

$$\overline{Q_l(t)}^{\,0} \equiv \int \mathrm{d}X\, W(0)\, \mathrm{e}^{O^\dagger t} Q_l = \bar{Q}_l(t)$$

and

$$\tilde{M}(t) = \left(\langle (O^\dagger Q) Q \rangle_{\mathrm{L}} + \int_0^\infty \mathrm{d}\tau\, \langle [O^\dagger G_{\mathrm{L}}(\tau)\, O^\dagger Q] Q \rangle_{\mathrm{L}} \right) * K^{(\mathrm{L})-1}\,. \tag{3.94}$$

For steady-state systems in which $\boldsymbol{\phi}^{(1)}(t) = \boldsymbol{\phi}^{(1)\mathrm{ss}}$, we find that the Q basis set and the local equilibrium averages are independent of time so that $\mathscr{P}_{\mathrm{L}}(t) = \mathscr{P}_{\mathrm{L}}^{\mathrm{ss}}$, and eq. (3.93) can be written as

$$0 = \overline{\mathrm{e}^{O^\dagger t} \mathscr{P}_{\mathrm{L}}^{\mathrm{ss}} O^\dagger Q_l}^{\,\mathrm{ss}} + \int_0^\infty \mathrm{d}\tau\, \overline{\mathrm{e}^{O^\dagger t} \mathscr{P}_{\mathrm{L}}^{\mathrm{ss}} O^\dagger G_{\mathrm{L}}(\tau)\, O^\dagger Q_l}^{\,\mathrm{ss}} = \langle (O^\dagger Q_l) Q \rangle_{\mathrm{L}} * K^{(\mathrm{L})-1} * \overline{Q(t)}^{\,\mathrm{ss}} + \int_0^\infty \mathrm{d}\tau\, \langle [O^\dagger G_{\mathrm{L}}(\tau)\, O^\dagger Q_l] Q \rangle_{\mathrm{L}} * K^{(\mathrm{L})-1} * \overline{Q(t)}^{\,\mathrm{ss}} \equiv \sum_{m=0}^{\infty} \tilde{M}_{lm}^{\mathrm{ss}} * \overline{Q_m(t)}^{\,\mathrm{ss}}, \tag{3.95}$$

where

$$\tilde{M}_{lm}^{\mathrm{ss}} = [\tilde{M}_{lm}(t)]_{\boldsymbol{\phi}^{(1)}(t) = \boldsymbol{\phi}^{(1)\mathrm{ss}}}\,.$$

Eq. (3.95) has been obtained by other researchers [12,14] in their work on steady states.

The N-ordering analysis of eq. (3.93) is straightforward since the local equilibrium average cumulants of correlation functions involving the Q basis set follow the same N-ordering rules as equilibrium cumulants of functions of the $Q^{(0)}$ basis set, where the $Q^{(0)}$ basis set is obtained from the Q set by setting $\boldsymbol{\phi}^{(1)}(t) = 0$. Thus we find

$$\tilde{M}_{l0}(t) \lesssim N\,, \qquad \tilde{M}_{lm}(t) \lesssim \begin{cases} 1 & \text{if } |l| \geq |m|\,, \\ N^{-|m|+|l|} & \text{if } |m| > |l|\,. \end{cases} \tag{3.96}$$

The N-ordering analysis of $\overline{Q(t)}^0 = \langle Q \rangle_t$ proceeds along similar lines of argument as the earlier analysis that lead to the conclusion $\langle Q_l \rangle_t = \langle Q_l \rangle_l$, and reveals that

$$\overline{Q_{2m}(t)}^0, \overline{Q_{2m+1}(t)}^0 \lesssim N^m\,, \tag{3.97}$$

and, for example,

$$\overline{Q_4(t)}^0 = \overline{Q_2(t)}^0\,\overline{Q_2(t)}^0 + \mathrm{O}(N)\,.$$

Combining (3.96) and (3.97) and neglecting terms of order $M = N(M/N)$ relative to order N terms, we obtain to order N

$$\begin{aligned}
&\dot{a}(\boldsymbol{r}_1, t) = \tilde{M}_{10}(\boldsymbol{r}_1, t)\,, \\
&\dot{\overline{Q_2(\boldsymbol{r}_1, \boldsymbol{r}_2)}}(t) = \tilde{M}_{20}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) + \tilde{M}_{22}(\boldsymbol{r}_1, \boldsymbol{r}_2; \boldsymbol{R}_1, \boldsymbol{R}_2, t) * \overline{Q_2(\boldsymbol{R}_1, \boldsymbol{R}_2)}(t)\,, \\
&\dot{\overline{Q_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)}}(t) = \tilde{M}_{30}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, t) \\
&\qquad + \tilde{M}_{32}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3; \boldsymbol{R}_1, \boldsymbol{R}_2, t) * \overline{Q_2(\boldsymbol{R}_1, \boldsymbol{R}_2)}(t) \\
&\qquad + \tilde{M}_{33}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3; \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, t) * \overline{Q_3(\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3)}(t) \\
&\qquad + \tilde{M}_{34}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3; \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{R}_4, t) * \overline{Q_4(\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{R}_4)}^*(t)\,,
\end{aligned}$$

where

$$\begin{aligned}
\overline{Q_4(\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{R}_4)}^*(t) &= \overline{Q_2(\boldsymbol{R}_1, \boldsymbol{R}_2)}(t)\,\overline{Q_2(\boldsymbol{R}_3, \boldsymbol{R}_4)}(t) \\
&\quad + \overline{Q_2(\boldsymbol{R}_1, \boldsymbol{R}_3)}(t)\,\overline{Q_2(\boldsymbol{R}_2, \boldsymbol{R}_4)}(t) + \overline{Q_2(\boldsymbol{R}_1, \boldsymbol{R}_4)}(t)\,\overline{Q_2(\boldsymbol{R}_2, \boldsymbol{R}_3)}(t)\,.
\end{aligned} \tag{3.98}$$

In the appendix we reexamine in detail these approximations in $\boldsymbol{r}$ space. It should be noted that

$$\frac{\partial \overline{Q_2(t)}^0}{\partial t} = \frac{\overline{\partial Q_2(t)}^0}{\partial t} = \overline{O^{\dagger} Q_2}(t)\,,$$

since $\partial Q_2(t)/\partial t$ is proportional to $Q_1(t)$ and $\overline{Q_1(t)}^0 = 0$.

Since the nonlinear operator $O^\dagger$ behaves linearly in the leading N order factorization of $\tilde{M}$, $\tilde{M}$ factors in the same way as the equilibrium hydrodynamic matrices M discussed in earlier work [6]. That is, if the spatial argument $\boldsymbol{r}_l$ is removed from the other arguments $\boldsymbol{r}_i$ in the matrix $\tilde{M}_{lm}$, then it factors to leading N order as

$$\begin{aligned} &\tilde{M}_{lm}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_l; \boldsymbol{R}_1, \ldots, \boldsymbol{R}_m) \\ &\quad = \tilde{M}_{l-1m-1}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{l-1}; \boldsymbol{R}_1, \ldots \boldsymbol{R}_{m-1})\, \delta(\boldsymbol{r}_l - \boldsymbol{R}_m)\, \delta_{a_l b_m} + \text{corrections}\,, \end{aligned} \tag{3.99}$$

where $\boldsymbol{R}_m$ is one of the $|m|$ spatial arguments of $\tilde{M}$ in the set m, and

$$\begin{aligned} l - 1 &\equiv \{a_1(\boldsymbol{r}_1), \ldots, a_{l-1}(\boldsymbol{r}_{l-1})\}\,, \\ m - 1 &\equiv \{b_1(\boldsymbol{R}_1), \ldots, b_{m-1}(\boldsymbol{R}_{m-1})\}\,. \end{aligned} \tag{3.100}$$

In Fourier space, the off-diagonal matrix M_{l-1m-1} is of leading N order $N^{2-|m|}$, whereas the correction terms are at least a factor of N lower in order. In particular, for $\tilde{M}_{22}(t)$ we find that to leading N order (when evaluated in Fourier space)

$$\begin{aligned} &\tilde{M}_{22}(a(\boldsymbol{r}_1), b(\boldsymbol{r}_2); c(\boldsymbol{R}_1), d(\boldsymbol{R}_2), t) = \tilde{M}_{11}(a(\boldsymbol{r}_1); c(\boldsymbol{R}_1), t)\, \delta_{bd}\delta(\boldsymbol{r}_2 - \boldsymbol{R}_2) \\ &\quad + \tilde{M}_{11}(a(\boldsymbol{r}_1); d(\boldsymbol{R}_2), t)\, \delta_{bc}\delta(\boldsymbol{r}_2 - \boldsymbol{R}_1) + \tilde{M}_{11}(b(\boldsymbol{r}_2); c(\boldsymbol{R}_1), t)\, \delta_{ad}\delta(\boldsymbol{r}_1 - \boldsymbol{R}_2) \\ &\quad + \tilde{M}_{11}(b(\boldsymbol{r}_2); d(\boldsymbol{R}_2), t)\, \delta_{ac}\delta(\boldsymbol{r}_1 - \boldsymbol{R}_1)\,, \end{aligned} \tag{3.101}$$

so to leading order in N

$$\begin{aligned} \overline{\dot{Q}_2(a(\boldsymbol{r}_1), b(\boldsymbol{r}_2), t)}^0 &= \tilde{M}_{20}(a(\boldsymbol{r}_1), b(\boldsymbol{r}_2), t) \\ &\quad + \int \mathrm{d}\boldsymbol{R}_1\, [\tilde{M}_{11}(\boldsymbol{r}_1; \boldsymbol{R}_1, t) \cdot \overline{Q_2(\boldsymbol{R}_1, \boldsymbol{r}_2, t)}^0]_{ab} \\ &\quad + \int \mathrm{d}\boldsymbol{R}_1\, [\tilde{M}_{11}(\boldsymbol{r}_2, \boldsymbol{R}_1, t) \cdot \overline{Q_2(\boldsymbol{R}_1, \boldsymbol{r}_1, t)}^0]_{ba}\,. \end{aligned} \tag{3.102}$$

In systems in which the gradients of the thermodynamic forces $\boldsymbol{\phi}^{(1)}(t)$ are small, $\tilde{M}_{11}(\boldsymbol{r}_1; \boldsymbol{R}_1, t)$ is strongly peaked around $\boldsymbol{R}_1 = \boldsymbol{r}_1$, and we may simplify (3.102) to obtain

$$\begin{aligned}\overline{\dot{Q}_2(a(\boldsymbol{r}_1), b(\boldsymbol{r}_2), t)}^0 = {} & [\tilde{M}_{11}(\boldsymbol{r}_1, t) \cdot \overline{Q_2(\boldsymbol{r}_1, \boldsymbol{r}_2, t)}^0]_{ab} \\ & + [\tilde{M}_{11}(\boldsymbol{r}_2, t) \cdot \overline{Q_2(\boldsymbol{r}_2, \boldsymbol{r}_1, t)}^0]_{ba} + \tilde{M}_{20}(a(\boldsymbol{r}_1), b(\boldsymbol{r}_2), t) \,. \end{aligned} \tag{3.103}$$

In wavevector space, eq. (3.103) is written as

$$\begin{aligned}\overline{Q_2(a(\boldsymbol{k}), b(\boldsymbol{k}'), t)}^0 = {} & \tilde{M}_{20}(a(\boldsymbol{k}), b(\boldsymbol{k}'), t) \\ & + \int \mathrm{d}\boldsymbol{r}_2 \exp(\mathrm{i}\boldsymbol{k}' \cdot \boldsymbol{r}_2)\, [\tilde{M}_{11}(\boldsymbol{r}_2) \cdot \overline{Q_2(\boldsymbol{r}_2, \boldsymbol{k}, t)}^0]_{ba} \\ & + \int \mathrm{d}\boldsymbol{r}_1 \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}_1)\, [\tilde{M}_{11}(\boldsymbol{r}_1, t) \cdot \overline{Q_2(\boldsymbol{r}_1, \boldsymbol{k}', t)}^0]_{ab} \,. \end{aligned} \tag{3.104}$$

For a steady-stage system characterized by linear shear flow and uniform internal energy and number densities, we find that the transport and thermodynamic properties of the system are uniform and therefore eq. (3.104) may be written as

$$0 = [\tilde{M}_{11}^{\mathrm{ss}} \cdot \overline{Q_2(\boldsymbol{k}, \boldsymbol{k}', t)}^0]_{ab} + [\tilde{M}_{11}^{\mathrm{ss}} \cdot \overline{Q_2(\boldsymbol{k}', \boldsymbol{k}, t)}^0]_{ba} + \tilde{M}_{20}^{\mathrm{ss}}(a(\boldsymbol{k}), b(\boldsymbol{k}')) \,, \tag{3.105}$$

where $\tilde{M}_{11}^{\mathrm{ss}}$ is an operator in $\boldsymbol{k}$ space. Similarly, to order N only the leading order factorizations of $\tilde{M}_{32}(t)$, $\tilde{M}_{33}(t)$ and $\tilde{M}_{34}(t)$ appear in the equation for $\overline{\dot{Q}_3(t)}^0$; these factorizations are

$$\begin{aligned} & \tilde{M}_{32}(t) = \tilde{M}_{21}(t) I_{11} + \mathrm{O}(1/N) \,, \qquad \tilde{M}_{33}(t) = \tilde{M}_{11}(t) I_{22} + \mathrm{O}(1/N) \,, \\ & \tilde{M}_{34}(t) = \tilde{M}_{12}(t) I_{22} + \mathrm{O}(1/N^2) \,, \end{aligned} \tag{3.106}$$

where I_{11} and I_{22} represent delta functions among the spatial arguments and the hydrodynamic indices of $\tilde{M}(t)$.

4. An alternative approach

It is also possible to obtain eq. (3.95) starting from the generalized equation of motion for the set Q. If we define the basis set C to be the basis set Q with the thermodynamic forces $\boldsymbol{\phi}^{(1)}(t)$ set equal to $\boldsymbol{\phi}^{(1)}(t')$, where t' is some arbitrary time, so that

$$C = (Q)_{\boldsymbol{\phi}^{(1)}(t)=\boldsymbol{\phi}^{(1)}(t')} ,$$

then according to eq. (2.19) of section 2, the equation of motion for C averaged over the conditional distribution $\tilde{\rho}$ for the internal modes is given by

$$\langle \dot{C}(X_{\mathrm{t}}, t)\rangle_{\mathrm{i}} = \langle K_C'(t)\rangle_{\mathrm{i}} + O^{\dagger}\langle C(X_{\mathrm{t}}, t)\rangle_{\mathrm{i}} , \tag{4.1}$$

where

$$K_C'(t) = -\mathrm{e}^{(1-\tilde{\mathscr{P}})Lt}(1 - \tilde{\mathscr{P}})LC$$

and

$$\tilde{P}B \equiv \int \mathrm{d}X_{\mathrm{i}}\, \tilde{\rho}B \equiv \int \mathrm{d}X_{\mathrm{i}}\, \rho_{\mathrm{i}}\, \mathrm{e}^{-\beta\phi}\, \mathrm{e}^{\beta\omega}B \equiv \langle B\rangle_{\mathrm{i}} .$$

Since C is independent of the internal modes and a function only of the translational phase space X_{t},

$$(1 - \tilde{\mathscr{P}})LC = \widehat{\boldsymbol{\nabla}_{r}{}^{N}}\, \phi \cdot \boldsymbol{\nabla}_{p^N} C .$$

Furthermore, since $K_C'(t) = (1 - \tilde{\mathscr{P}})K_C'(t)$ by construction, $\langle K_C'(t)\rangle_i = 0$, and hence

$$\langle \dot{C}(X_{\mathrm{t}}, t)\rangle_{\mathrm{i}} = O^{\dagger}\langle C(X_{\mathrm{t}}, t)\rangle_{\mathrm{i}} , \tag{4.2}$$

which implies that

$$\langle C(X_{\mathrm{t}}, t)\rangle_{\mathrm{i}} = \mathrm{e}^{O^{\dagger}t}C(X_{\mathrm{t}}) \tag{4.3}$$

and

$$\langle \dot{C}(X_{\mathrm{t}}, t)\rangle_{\mathrm{i}} = O^{\dagger}\, \mathrm{e}^{O^{\dagger}t}C(X_{\mathrm{t}}) . \tag{4.4}$$

Henceforth, for notational simplicity, in subsequent equations we shall omit the $\langle\ \rangle_{\mathrm{i}}$ brackets which denote the average of the internal degrees of freedom over the conditional distribution $\tilde{\rho}$ for the bath.

Defining the time independent projection operator $\mathscr{P}_{\mathrm{L}}^{(\prime)}$ by

$$\mathscr{P}_{\mathrm{L}}^{(\prime)}B = \langle BC\rangle_{\mathrm{L}}^{(\prime)} * K^{(\prime)-1} * C ,$$

where $K^{(')} \equiv \langle CC \rangle_{\mathrm{L}}^{(')}$ and $\langle B \rangle_{\mathrm{L}}^{(')}$ represents the average of B over the local equilibrium distribution function $\sigma_1(t')$, and using the well-known operator identity

$$\mathrm{e}^{(A+B)t} = \mathrm{e}^{At} + \int_0^t \mathrm{d}\tau\, \mathrm{e}^{A(t-\tau)} B\, \mathrm{e}^{(A+B)\tau} , \tag{4.5}$$

we obtain [13]

$$\dot{C}(t) = K_C(t) + \hat{M} * C(t) , \tag{4.6}$$

where $K_C(t)$ and $\hat{M}$ are defined by

$$K_C(t) = \exp[(1 - \mathcal{P}_{\mathrm{L}}^{(')})O^\dagger t]\,(1 - \mathcal{P}_{\mathrm{L}}^{(')})O^\dagger C ,$$

$$\hat{M} = \Big(\langle (O^\dagger C) C \rangle_{\mathrm{L}}^{(')} + \int_0^\infty \mathrm{d}\tau\, \langle [O^\dagger G_{\mathrm{L}}^{(')}(\tau)\, O^\dagger C] C \rangle_{\mathrm{L}}^{(')} \Big) * K^{(')-1} . \tag{4.7}$$

If we assume that the initial distribution function for the translational degrees of freedom $W(0)$ is an analytic function of the hydrodynamic densities,

$$W(0) = f(A(0)) ,$$

then

$$\int \mathrm{d}X_t\, W(0)\, K_C(t) \equiv \overline{K_C(t)}^0 = 0$$

by construction. It should be emphasized that the notation $\overline{B(t)}^0$ denotes the nonequilibrium average of $B(t)$ over the initial distribution function $W(0)$. Thus, if we average (4.6) over the initial distribution function $W(0)$, we obtain

$$\overline{\dot{C}(t)}^0 = \hat{M} * \overline{C(t)}^0 . \tag{4.8}$$

If $W(0)$ is the steady-state distribution W^{ss}, then if $\boldsymbol{\phi}^{(1)}(t') = \boldsymbol{\phi}^{(1)\mathrm{ss}}$ we obtain $C = Q$ and $\hat{M} = \tilde{M}$, which implies

$$\tilde{M}^{\mathrm{ss}} * \overline{Q(t)}^{\mathrm{ss}} = 0 , \tag{4.9}$$

which is the result we derived in eq. (3.95) of section 3 for steady-state systems.

Eq. (4.8) is deceptive in that it really does not depend on the thermodynamic forces $\boldsymbol{\phi}^{(1)}(t')$. To establish this fact, we take the functional derivative of (4.8) with respect to $\boldsymbol{\phi}^{(1)}(t')$, and find that

$$\overline{O^{\dagger}\,\mathrm{e}^{O^{\dagger}t}\frac{\delta C_l}{\delta\boldsymbol{\phi}^{(1)}(t')}}^{\,0}=\sum_{m=0}^{\infty}\frac{\delta\hat{M}_{lm}}{\delta\boldsymbol{\phi}^{(1)}(t')}*\overline{C_m(t)}^{\,0}+\sum_{m=0}^{\infty}\hat{M}_{lm}*\overline{\mathrm{e}^{O^{\dagger}t}\frac{\delta C_m}{\delta\boldsymbol{\phi}^{(1)}(t')}}^{\,0}\,. \tag{4.10}$$

Since

$$\begin{aligned}\frac{\delta}{\delta\boldsymbol{\phi}^{(1)}(t')}\left(\langle BC\rangle_{\mathrm{L}}^{(\prime)}*K^{(\prime)-1}\right)&=\left\langle\frac{\delta B}{\delta\boldsymbol{\phi}^{(1)}(t')}\,C\right\rangle_{\mathrm{L}}^{(\prime)}*K^{(\prime)-1}\\&\quad-\langle BC\rangle_{\mathrm{L}}^{(\prime)}*K^{(\prime)-1}*\left\langle\frac{\delta C}{\delta\boldsymbol{\phi}^{(1)}(t')}\,C\right\rangle_{\mathrm{L}}^{(\prime)}*K^{(\prime)-1}\end{aligned}$$

and

$$\frac{\delta C_l}{\delta\boldsymbol{\phi}^{(1)}(t')}=\left\langle\frac{\delta C_l}{\delta\boldsymbol{\phi}^{(1)}(t')}\,C_{\hat{m}}\right\rangle_{\mathrm{L}}^{(\prime)}*K_{\hat{m}m}^{(\prime)-1}*C_m\,, \tag{4.11}$$

where $|\hat{m}|=|m|=|l|-1$, we find that

$$\frac{\delta\mathscr{P}_{\mathrm{L}}^{(\prime)}B}{\delta\boldsymbol{\phi}^{(1)}(t')}=\mathscr{P}_{\mathrm{L}}^{(\prime)}\frac{\delta B}{\delta\boldsymbol{\phi}^{(1)}(t')}\,, \tag{4.12}$$

and for $|l|\geqslant 1$

$$\begin{aligned}\frac{\delta\hat{M}_{lm}}{\delta\boldsymbol{\phi}^{(1)}(t')}&=\left\langle\frac{\delta C_l}{\delta\boldsymbol{\phi}^{(1)}(t')}\,C_{\hat{j}}\right\rangle_{\mathrm{L}}^{(\prime)}*K_{\hat{j}j}^{(\prime)-1}*\hat{M}_{jm}\\&\quad-\hat{M}_{lm+1}*\left\langle\frac{\delta C_{m+1}}{\delta\boldsymbol{\phi}^{(1)}(t')}\,C_{\hat{m}}\right\rangle_{\mathrm{L}}^{(\prime)}*K_{\hat{m}m}^{(\prime)-1}\,,\end{aligned} \tag{4.13}$$

where $|\hat{j}|=|j|=|l|-1$ and $|m+1|=|m|+1$. Inserting (4.11), (4.12) and (4.13) into eq. (4.10) yields

$$\frac{\delta}{\delta\boldsymbol{\phi}^{(1)}(t')}\left(\overline{O^{\dagger}C_l(t)}^{\,0}-\sum_{m=0}^{\infty}\hat{M}_{lm}*\overline{C_m(t)}^{\,0}\right)=0\,, \tag{4.14}$$

so we see that in fact (4.8) is independent of $\boldsymbol{\phi}^{(1)}(t')$ and must therefore hold for *any* choice of $\boldsymbol{\phi}^{(1)}(t')$. For example, selecting $\boldsymbol{\phi}^{(1)}(t')=\boldsymbol{\phi}^{(1)}(t)$ yields $C=Q$, and eq. (4.8) is then

$$\overline{O^{\dagger}Q_l(t)}^{0} = \sum_{m=0}^{\infty} \tilde{M}_{lm}(t) * \overline{Q_m(t)}^{0} , \tag{4.15}$$

which is just eq. (3.93) of the previous section. Furthermore, setting $\boldsymbol{\phi}^{(1)}(t') = \boldsymbol{\phi}^{(1)\mathrm{ss}}$ yields eq. (3.95). If we set $\boldsymbol{\phi}^{(1)}(t') = 0$ so that $C = Q^{(0)}$, where $Q^{(0)}$ is the multilinear basis set defined with respect to the equilibrium distribution function, we obtain the interesting equation

$$\overline{O^{\dagger}Q_l^{(0)}(t)}^{0} = \sum_{m=0}^{\infty} \tilde{M}_{lm}(t) * \overline{Q_m^{(0)}(t)}^{0} . \tag{4.16}$$

Since (4.16) is just eq. (4.15) in disguise, we can use the fact that

$$\begin{aligned}\left(\frac{\delta \tilde{M}_{lm}(t)}{\delta \boldsymbol{\phi}^{(1)}(t)}\right)_{\boldsymbol{\phi}^{(1)}(t)=0} &= \left(\left\langle \frac{\delta Q_l}{\delta \boldsymbol{\phi}^{(1)}(t)}\, Q_{\hat{j}} \right\rangle_{\mathrm{L}}\right)_{\boldsymbol{\phi}^{(1)}(t)=0} * K_{\hat{j}j}^{-1} * M_{jm} \\ &\quad - M_{lm+1} * \left(\left\langle \frac{\delta Q_{m+1}}{\delta \boldsymbol{\phi}^{(1)}(t)}\, Q_{\hat{m}} \right\rangle_{\mathrm{L}}\right)_{\boldsymbol{\phi}^{(1)}(t)=0} * K_{\hat{m}m}^{-1} ,\end{aligned} \tag{4.17}$$

to write M_{lm} in terms of the $\boldsymbol{\phi}^{(1)}(t) = 0$ values of functional derivatives of $\tilde{M}(t)$, and obtain the result that for $l = 1$,

$$\begin{aligned}\dot{a}(\boldsymbol{r}_1, t) &= \overline{O^{\dagger}A(\boldsymbol{r}_1, t)}^{0} \\ &= \sum_{n=0}^{\infty} \left(\frac{\delta^n \tilde{M}_{10}(\boldsymbol{r}_1, t)}{n!\, \Pi_{i=1}^{n}\, \delta\langle A(\boldsymbol{R}_i)\rangle_{\mathrm{L}}}\right)_{\boldsymbol{\phi}^{(1)}(t)=0} \cdot a(\boldsymbol{R}_1, t) \cdots a(\boldsymbol{R}_n, t) ,\end{aligned} \tag{4.18}$$

where $a(\boldsymbol{R}_i, t) \equiv \langle A(\boldsymbol{R}_i)\rangle_{\mathrm{L}} - \langle A(\boldsymbol{R}_i)\rangle$. Kavassalis and Oppenheim [13] obtained eq. (4.18) for ordinary fluid systems and showed that when the gradients of the thermodynamic forces are small, each derivative of $\tilde{M}_{10}(\boldsymbol{r}_1, t)$ with respect to $a(\boldsymbol{R}_i, t)$ is sharply peaked around $\boldsymbol{R}_i = \boldsymbol{r}_1$, so that

$$\frac{\delta^n \tilde{M}_{10}(\boldsymbol{r}_1, t)}{n!\, \Pi_{i=1}^{n}\, \delta\langle A(\boldsymbol{R}_i)\rangle_{\mathrm{L}}} \approx \frac{V^n \delta^n \tilde{M}_{10}(\boldsymbol{r}_1, t)}{n!\, \delta\langle A\rangle_{\mathrm{H}}(\boldsymbol{r}_1, t)^n} \prod_{i=1}^{n} \delta(\boldsymbol{R}_i - \boldsymbol{r}_1) . \tag{4.19}$$

They then showed that the time and space dependence of the pressure and transport coefficients which appear in $\tilde{M}_{10}(\boldsymbol{r}_1, t)$ arises due to their functional dependence on the nonequilibrium mass and internal energy densities. When the scale on which the thermodynamic forces vary is long compared to the correlation length of the system in equilibrium, the functional dependence of the nonequilibrium pressure and transport coefficients on the mass and energy densities is the same as that in equilibrium.

If we extend this analysis to the equation of motion for the bilinear variables, we find that for systems with small gradients

$$\tilde{M}_{20}(\boldsymbol{r}_1, \boldsymbol{r}_2, t) = \sum_{n=0}^{\infty} \left(\frac{V^n\, \delta^n \tilde{M}_{20}(\boldsymbol{r}_1, \boldsymbol{r}_2, t)}{n!\delta\langle A\rangle_{\mathrm{H}}(\boldsymbol{R}, t)^n} \right)_{\boldsymbol{\phi}^{(1)}(t)=0} \cdot a(\boldsymbol{R}, t)^n \,,$$

$$\tilde{M}_{11}(\boldsymbol{r}_1, \boldsymbol{r}, t) = \frac{\delta \tilde{M}_{10}(\boldsymbol{r}_1, t)}{\delta a(\boldsymbol{r}, t)} = \frac{\delta \dot{a}(\boldsymbol{r}_1, t)}{\delta a(\boldsymbol{r}_1, t)} \delta(\boldsymbol{r}_1 - \boldsymbol{r}) \,, \tag{4.20}$$

where $\boldsymbol{R} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)$, and hence we conclude

$$\dot{a}(\boldsymbol{r}_1, t) = \tilde{M}_{10}(\boldsymbol{r}_1, t) \,, \tag{4.21}$$

$$\begin{aligned} \overset{\cdot}{\overline{Q_2(a(\boldsymbol{r}_1), b(\boldsymbol{r}_2), t)}}^{0} &= \left(\frac{\delta \dot{a}(\boldsymbol{r}_1, t)}{\delta a(\boldsymbol{r}_1, t)} \cdot \overline{Q_2(\boldsymbol{r}_1, \boldsymbol{r}_2, t)}^{0} \right)_{ab} \\ &\quad + \left(\frac{\delta \dot{a}(\boldsymbol{r}_2, t)}{\delta a(\boldsymbol{r}_2, t)} \cdot \overline{Q_2(\boldsymbol{r}_2, \boldsymbol{r}_1, t)}^{0} \right)_{ba} + \tilde{M}_{20}(a(\boldsymbol{r}_1), b(\boldsymbol{r}_2), t) \,. \end{aligned} \tag{4.22}$$

In eq. (4.22), the functional dependence of the nonequilibrium thermodynamic and transport coefficients which appear in $\tilde{M}_{20}(t)$ and $\dot{a}(t)$ on the nonequilibrium mass and internal energy densities is the same as that in equilibrium. For steady-state systems linearly displaced from equilibrium, we may write (4.22) to linear order in $\boldsymbol{\phi}^{(2)}$ as

$$\begin{aligned} 0 = \tilde{M}_{20}^{\mathrm{ss}}(a(\boldsymbol{r}_1), b(\boldsymbol{r}_2)) &+ \left(\frac{\delta \dot{a}_a(\boldsymbol{r}_1)}{\delta \boldsymbol{\phi}_i^{(1)}(\boldsymbol{r}_1)} \langle (\hat{A}_{\mathrm{L}})_b(\boldsymbol{r}_2)\, [\hat{A}_{\mathrm{L}}(\boldsymbol{r}_2)]_j \rangle_{\mathrm{L}} \right)^{\mathrm{ss}} \boldsymbol{\phi}_{ij}^{(2)}(\boldsymbol{r}_1, \boldsymbol{r}_2) \\ &+ \left(\frac{\delta \dot{a}_b(\boldsymbol{r}_2)}{\delta \boldsymbol{\phi}_i^{(1)}(\boldsymbol{r}_2)} \langle (\hat{A}_{\mathrm{L}})_a(\boldsymbol{r}_1)\, [\hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)]_j \rangle_{\mathrm{L}} \right)^{\mathrm{ss}} \boldsymbol{\phi}_{ij}^{(2)}(\boldsymbol{r}_2, \boldsymbol{r}_1) \,, \end{aligned} \tag{4.23}$$

where i and j are summed over the hydrodynamic densities. In the next two sections, we shall analyze these equations more closely for specific systems.

5. Granular flow hydrodynamics

In this section we present the nonlinear hydrodynamic equations for the granular flow system and analyze the linearized form of these equations in various regimes reflecting the degree of inelasticity of each collision between granular particles. The solutions of the linear equations are contrasted with those for ordinary hydrodynamic system in which the particles interact elastically.

According to eq. (3.91), the time dependence of the nonequilibrium averages of the hydrodynamic densities is governed by the equation

$$\dot{a}(\boldsymbol{r},t) = \langle O^{\dagger} A(\boldsymbol{r})\rangle_{\mathrm{L}} + \int_{0}^{\infty} \mathrm{d}\tau \, \langle \{\exp[\mathcal{Q}_{\mathrm{L}}(t)\, O^{\dagger}\tau]\, \mathcal{Q}_{\mathrm{L}}(t)\, O^{\dagger} A(\boldsymbol{r})\} \mathcal{Q}_{\mathrm{L}}(t)\, \psi_{\mathrm{L}}(t)\rangle_{\mathrm{L}} \,, \tag{5.1}$$

where $\psi_{\mathrm{L}}(t)$ is defined by

$$O\sigma_{1}(t) = \psi_{\mathrm{L}}(t)\, \sigma_{1}(t) \,, \tag{5.2}$$

and

$$a(\boldsymbol{r},t) \equiv \langle A(\boldsymbol{r})\rangle_{\mathrm{L}} - \langle A(\boldsymbol{r})\rangle = \frac{1}{V} \langle \hat{A}\rangle_{\mathrm{H}}(\boldsymbol{r},t) \,.$$

In ordinary hydrodynamics, the time derivative of the hydrodynamic variables which compose A are proportional to the gradients of the system and are therefore small for relatively uniform systems. For granular flow systems, however, the energy is not conserved and hence the time derivative of the energy density has terms which are not proportional to a gradient and not necessarily small, even for uniform systems. Therefore, we shall restrict our attention to cases when the energy density is slowly varying over the timescale τ_m. This assumption limits the magnitude of γ_{jk} in the system evolution operators Ω_1 and Ω_2. To indicate this restriction, we will associate a factor of the small parameter λ with each factor of γ_{jk}.

Following the development of the ordinary hydrodynamic equations [11], we introduce the fourth rank tensor $\boldsymbol{O}_{\mathrm{p}}(\boldsymbol{r},t)$ and the second rank tensor $\boldsymbol{O}_{\mathrm{e}}(\boldsymbol{r},t)$ which are defined by

$$[\boldsymbol{O}_{\mathrm{p}}(\boldsymbol{r},t)]_{ijkl} = \int_{0}^{\infty} \mathrm{d}\tau \, \frac{\langle [G_{\mathrm{L}}(\tau)\, \boldsymbol{\tau}_{ij}^{+}] \mathcal{Q}_{\mathrm{L}}(t)\, \boldsymbol{\tau}_{kl}^{+}\rangle_{\mathrm{H}}(\boldsymbol{r},t)}{V}$$

$$= \frac{1}{\beta(\boldsymbol{r},t)} \{[\zeta(\boldsymbol{r},t) - \tfrac{2}{3}\eta(\boldsymbol{r},t)]\delta_{ij}\delta_{kl} + \eta(\boldsymbol{r},t)\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\} \,,$$

$$[\boldsymbol{O}_{\mathrm{e}}(\boldsymbol{r}, t)]_{ij} = \int_0^\infty \mathrm{d}\tau \frac{\langle [G_{\mathrm{L}}(\tau)\, (\boldsymbol{J}_{\mathrm{E}}^{+})_i] \mathcal{Q}_{\mathrm{L}}(t)\, (\boldsymbol{J}_{\mathrm{E}}^{+})_j \rangle_{\mathrm{H}}(\boldsymbol{r}, t)}{V}$$

$$= \frac{1}{K_{\mathrm{B}} \beta^2(\boldsymbol{r}, t)} \kappa(\boldsymbol{r}, t)\, \delta_{ij} \,, \tag{5.3}$$

where

$$\boldsymbol{\tau}^{+}(\boldsymbol{r}) = \sum_j \left(\frac{\boldsymbol{p}_j^{+} \boldsymbol{p}_j^{+}}{m} + \tfrac{1}{2} \sum_{k \neq j} \boldsymbol{r}_{jk} \boldsymbol{F}_{jk} \right) \delta(\boldsymbol{r} - \boldsymbol{r}_j) \,,$$

$$\boldsymbol{J}_{\mathrm{E}}^{+}(\boldsymbol{r}) = \sum_j \left(\frac{e_j \boldsymbol{p}_j^{+}}{m} + \frac{1}{2m} \sum_{k \neq j} \boldsymbol{r}_{jk} \boldsymbol{p}_j^{+} \cdot \boldsymbol{F}_{jk} \right) \delta(\boldsymbol{r} - \boldsymbol{r}_j) \,. \tag{5.4}$$

In eq. (5.3), K_{B} is Boltzmann's constant and $\zeta(\boldsymbol{r}, t)$, $\eta(\boldsymbol{r}, t)$ and $\kappa(\boldsymbol{r}, t)$ are, respectively, the bulk viscosity, the shear viscosity and the thermal conductivity of a homogeneous system with number density $n(\boldsymbol{r}, t)$ and internal energy density $e^{+}(\boldsymbol{r}, t)$. Evaluating eq. (5.1) explicitly up to second order in λ and gradients of the system, yields [9]

$$\dot{n}(\boldsymbol{r}, t) = -\boldsymbol{\nabla}_r \cdot [n(\boldsymbol{r}, t)\, \boldsymbol{v}(\boldsymbol{r}, t)] \,, \tag{5.5}$$

$$\begin{aligned}
\overset{\bullet}{[e^{+}(\boldsymbol{r}, t) + \tfrac{1}{2} m v^2(\boldsymbol{r}, t)\, n(\boldsymbol{r}, t)]} &= -\boldsymbol{\nabla}_r \cdot \{\boldsymbol{v}(\boldsymbol{r}, t)\, [h^{+}(\boldsymbol{r}, t) + \tfrac{1}{2} m n(\boldsymbol{r}, t)\, v^2(\boldsymbol{r}, t)]\} \\
&+ \boldsymbol{\nabla}_r \cdot [\kappa(\boldsymbol{r}, t)\, \boldsymbol{\nabla}_r T(\boldsymbol{r}, t)] + \boldsymbol{\nabla}_r \cdot \{\boldsymbol{v}(\boldsymbol{r}, t)\, [\zeta(\boldsymbol{r}, t) - \tfrac{2}{3} \eta(\boldsymbol{r}, t)] \boldsymbol{\nabla}_r \cdot \boldsymbol{v}(\boldsymbol{r}, t)\} \\
&+ \boldsymbol{\nabla}_{r_i} [\boldsymbol{v}_j(\boldsymbol{r}, t)\, \eta(\boldsymbol{r}, t)\, \boldsymbol{\nabla}_{r_i} \boldsymbol{v}_j(\boldsymbol{r}, t) + \boldsymbol{v}_j(\boldsymbol{r}, t)\, \eta(\boldsymbol{r}, t)\, \boldsymbol{\nabla}_{r_j} \boldsymbol{v}_i(\boldsymbol{r}, t)] \\
&+ \lambda \boldsymbol{\nabla}_r \cdot \{\boldsymbol{v}(\boldsymbol{r}, t)\, \Theta_1(\boldsymbol{r}, t)\, [\beta - \beta(\boldsymbol{r}, t)]\} + \lambda [\Theta_2(\boldsymbol{r}, t)\, \beta(\boldsymbol{r}, t)\, \boldsymbol{\nabla}_r \cdot \boldsymbol{v}(\boldsymbol{r}, t)] \\
&+ \lambda^2 \Lambda_2(\boldsymbol{r}, t)\, [\beta - \beta(\boldsymbol{r}, t)] - \lambda \frac{\Lambda_1(\boldsymbol{r}, t)}{m} \left(\frac{\beta}{\beta(\boldsymbol{r}, t)} - 1 \right) ,
\end{aligned}$$

$$\begin{aligned}
\overset{\bullet}{[m\boldsymbol{v}(\boldsymbol{r}, t)\, n(\boldsymbol{r}, t)]} = -\boldsymbol{\nabla}_r P_{\mathrm{h}}(\boldsymbol{r}, t) &- \boldsymbol{\nabla}_r \cdot [n(\boldsymbol{r}, t)\, m\boldsymbol{v}(\boldsymbol{r}, t)\, \boldsymbol{v}(\boldsymbol{r}, t)] \\
&+ \boldsymbol{\nabla}_r \{[\zeta(\boldsymbol{r}, t) - \tfrac{2}{3} \eta(\boldsymbol{r}, t)] \boldsymbol{\nabla}_r \cdot \boldsymbol{v}(\boldsymbol{r}, t)\} + \boldsymbol{\nabla}_{r_i} [\eta(\boldsymbol{r}, t)\, \boldsymbol{\nabla}_{r_i} \boldsymbol{v}(\boldsymbol{r}, t)] \\
&+ \boldsymbol{\nabla}_{r_i} [\eta(\boldsymbol{r}, t)\, \boldsymbol{\nabla}_r \boldsymbol{v}_i(\boldsymbol{r}, t)] + \lambda \boldsymbol{\nabla}_r \Theta_1(\boldsymbol{r}, t)\, [\beta - \beta(\boldsymbol{r}, t)] \,,
\end{aligned}$$

$$\tag{5.6}$$

where

$$V\Theta_1(\boldsymbol{r}, t) = \int_0^\infty \mathrm{d}\tau\, \langle \boldsymbol{\tau}^{+\mathrm{d}}(\tau)\, Y^{\mathrm{d}} \rangle_{\mathrm{H}}(\boldsymbol{r}, t)\,,$$

$$V\Theta_2(\boldsymbol{r}, t) = \int_0^\infty \mathrm{d}\tau\, \langle X^{\mathrm{d}}(\tau)\, \boldsymbol{\tau}^{+\mathrm{d}} \rangle_{\mathrm{H}}(\boldsymbol{r}, t)\,,$$

$$V\Lambda_1(\boldsymbol{r}, t) = \Big\langle \sum_j \sum_{k \neq j} \gamma_{jk} \Big\rangle_{\mathrm{H}}(\boldsymbol{r}, t)\,, \qquad V\Lambda_2(\boldsymbol{r}, t) = \int_0^\infty \mathrm{d}\tau\, \langle X^{\mathrm{d}}(\tau)\, Y^{\mathrm{d}} \rangle_{\mathrm{H}}(\boldsymbol{r}, t)\,, \tag{5.7}$$

i and j are summed from 1 to 3 and for an arbitrary dynamical variable B, $B^{\mathrm{d}}(\tau)$ is defined by $B^{\mathrm{d}}(\tau) \equiv G_{\mathrm{L}}(\tau)\, B$. To obtain eq. (5.6), we have used the facts that

$$\frac{\langle \tilde{\boldsymbol{\tau}}^{+} \rangle_{\mathrm{H}}(\boldsymbol{r}, t)}{V} = P_{\mathrm{h}}(\boldsymbol{r}, t)\, \boldsymbol{I}\,,$$

where $P_{\mathrm{h}}(\boldsymbol{r}, t)$ is the hydrostatic pressure, $\langle \boldsymbol{J}_{\mathrm{E}}^{+} \rangle_{\mathrm{H}}(\boldsymbol{r}, t) = 0$, and $h^{+}(\boldsymbol{r}, t) = e^{+}(\boldsymbol{r}, t) + P_{\mathrm{h}}(\boldsymbol{r}, t)$ is the internal enthalpy density. Note that unlike ordinary hydrodynamics in which $\dot{e}(\boldsymbol{r}, t)$ is proportional to a gradient, $\dot{e}(\boldsymbol{r}, t)$ has a term which exists for uniform systems, and reflects the fact that energy flows out of the system.

If $\lambda^{1/2}$ is on the order of the gradients of the system, the momentum density equation for the granular flow system is identical to the ordinary momentum density equation to second order in the gradients of the system, and the energy density equation contains only the additional term

$$-\frac{\lambda \langle \Sigma_j \Sigma_{k \neq j} \gamma_{jk} \rangle_{\mathrm{H}}(\boldsymbol{r}, t)}{mV} \left(\frac{\beta}{\beta(\boldsymbol{r}, t)} - 1 \right),$$

which accounts for the unidirectional flow of energy from the translational degrees of freedom in the granular flow system into the granular particle internal modes when $T(\boldsymbol{r}, t) > T$.

If we define the linear deviations of the hydrodynamic variables to be

$$\delta n(\boldsymbol{r}, t) = n(\boldsymbol{r}, t) - n_0\,, \qquad \delta \boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{v}(\boldsymbol{r}, t) - \boldsymbol{v}_0 = \boldsymbol{v}(\boldsymbol{r}, t)\,,$$
$$\delta e(\boldsymbol{r}, t) = e^{+}(\boldsymbol{r}, t) - e_0^{+}\,, \tag{5.8}$$

where n_0 and e_0^{+} are the equilibrium number and the internal energy densities respectively, then to linear order in the deviations we obtain

$$\begin{aligned}
&\dot{\delta} n(\boldsymbol{r},t) = -n_0 \nabla_r \cdot \delta \boldsymbol{v}(\boldsymbol{r},t)\,,\\
&mn_0\, \dot{\delta} \boldsymbol{v}(\boldsymbol{r},t) = -a\nabla_r\, \delta n(\boldsymbol{r},t) - b\nabla_r\, \delta e(\boldsymbol{r},t) + (\zeta + \tfrac{1}{3}\eta)\nabla_r\nabla_r \cdot \delta \boldsymbol{v}(\boldsymbol{r},t)\\
&\qquad\qquad + \eta\nabla^2\, \delta \boldsymbol{v}(\boldsymbol{r},t)\,,\\
&\dot{\delta} e(\boldsymbol{r},t) = -c\nabla_r \cdot \delta \boldsymbol{v}(\boldsymbol{r},t) + \kappa\left(\frac{\partial T}{\partial n}\right)_{e^+} \nabla_r \cdot \nabla_r\, \delta n(\boldsymbol{r},t)\\
&\qquad\qquad + \kappa\left(\frac{\partial T}{\partial e^+}\right)_n \nabla_r \cdot \nabla_r\, \delta e(\boldsymbol{r},t) - f\, \delta n(\boldsymbol{r},t) - g\, \delta e(\boldsymbol{r},t)\,,
\end{aligned} \tag{5.9}$$

where

$$a \equiv \chi_n - \frac{\lambda\Theta_1}{K_B T^2}\left(\frac{\partial T}{\partial n}\right)_{e^+}, \qquad b \equiv \chi_e - \frac{\lambda\Theta_1}{K_B T^2}\left(\frac{\partial T}{\partial e^+}\right)_n, \qquad c \equiv h^+ - \frac{\lambda\Theta_2}{K_B T},$$

$$f \equiv \frac{\lambda\Lambda_1}{mT}\left(\frac{\partial T}{\partial n}\right)_{e^+} - \frac{\lambda^2\Lambda_2}{K_B T^2}\left(\frac{\partial T}{\partial n}\right)_{e^+}, \qquad g \equiv \frac{\lambda\Lambda_1}{mT}\left(\frac{\partial T}{\partial e^+}\right)_n - \frac{\lambda^2\Lambda_2}{K_B T^2}\left(\frac{\partial T}{\partial e^+}\right)_n,$$

and

$$\chi_n = \left(\frac{\partial P}{\partial n}\right)_{e^+}, \qquad \chi_e = \left(\frac{\partial P}{\partial e^+}\right)_n. \tag{5.10}$$

These equations may be written in Fourier space as

$$\begin{aligned}
&\dot{\delta} n(k,t) = \mathrm{i}n_0 k\, \delta \boldsymbol{v}_\mathrm{l}(k,t)\,,\\
&\dot{\delta} e(k,t) = \mathrm{i}ck\, \delta \boldsymbol{v}_\mathrm{l}(k,t) - \left[\kappa\left(\frac{\partial T}{\partial n}\right)_{e^+} k^2 + f\right] \delta n(k,t)\\
&\qquad\qquad - \left[\kappa\left(\frac{\partial T}{\partial e^+}\right)_n k^2 + g\right] \delta e(k,t)\,,\\
&\dot{\delta} \boldsymbol{v}_\ell(k,t) = \frac{\mathrm{i}ak}{mn_0}\, \delta n(k,t) + \frac{\mathrm{i}bk}{mn_0}\, \delta e(k,t) - (\zeta + \tfrac{4}{3}\eta)\frac{k^2}{mn_0}\, \delta \boldsymbol{v}_\ell(k,t)\,,\\
&\dot{\delta} \boldsymbol{v}_\mathrm{t}(k,t) = -\frac{\eta}{mn_0} k^2\, \delta \boldsymbol{v}_\mathrm{t}(k,t) \equiv -\nu k^2\, \delta \boldsymbol{v}_\mathrm{t}(k,t)\,,
\end{aligned} \tag{5.11}$$

where we have defined the longitudinal and transverse velocities by

$$\delta \boldsymbol{v}_\ell(k,t) = \hat{\boldsymbol{k}} \cdot \delta \boldsymbol{v}(k,t)\,, \qquad \delta \boldsymbol{v}_\mathrm{t}(k,t) = \hat{\boldsymbol{k}} \wedge \delta \boldsymbol{v}(k,t)\,. \tag{5.12}$$

Note that the linearized equation of motion for the transverse velocity $\delta \boldsymbol{v}_\mathrm{t}(\boldsymbol{r},t)$ is uncoupled from the other hydrodynamic densities and is identical to that obtained for ordinary hydrodynamic systems. From eq. (5.11), we conclude that

$$\delta \boldsymbol{v}_\mathrm{t}(k,t) = \exp(-\nu k^2 t)\, \delta \boldsymbol{v}_\mathrm{t}(k)\,. \tag{5.13}$$

If we set λ to zero in eq. (5.11), the ordinary hydrodynamic equations are obtained. The eigenmodes or hydrodynamic modes of this system of equations are given to lowest order in k by

$$\begin{aligned} &T(k) = \delta e(k) - \frac{h^+}{n_0}\delta n(k)\,, \qquad S^+(k) = \frac{\chi_n}{c_0}\delta n(k) + \frac{\chi_e}{c_0}\delta e(k) + m\,\delta v_\ell(k)\,,\\ &S^-(k) = \frac{\chi_n}{c_0}\delta n(k) + \frac{\chi_e}{c_0}\delta e(k) - m\,\delta \boldsymbol{v}_\ell(k)\,, \end{aligned} \tag{5.14}$$

where c_0 is the zero frequency adiabatic sound velocity defined by

$$c_0^2 = \frac{\chi_n}{m} + \frac{\chi_e h^+}{m n_0}\,. \tag{5.15}$$

The time dependence of the hydrodynamic modes is given by

$$\begin{aligned} &T(k,t) = \exp(-\Gamma_\mathrm{T} k^2 t)\, T(k)\,,\\ &S^+(k,t) = \exp[(\mathrm{i}c_0 k - \Gamma_\mathrm{S} k^2)t]\, S^+(k)\,,\\ &S^-(k,t) = \exp[(-\mathrm{i}c_0 k - \Gamma_\mathrm{S} k^2)t]\, S^-(k)\,, \end{aligned} \tag{5.16}$$

where the sound attenuation and thermal diffusivity coefficients are

$$\begin{aligned} &\Gamma_\mathrm{S} = \tfrac{1}{2}\left[\lambda_\mathrm{H}\left(\frac{\partial T}{\partial e^+}\right)_n + \frac{\zeta + \frac{4}{3}\eta}{m n_0} - \Gamma_\mathrm{T}\right],\\ &\Gamma_\mathrm{T} = \frac{\lambda_\mathrm{H}}{m c_0^2}\left[\chi_n\left(\frac{\partial T}{\partial e^+}\right)_n - \chi_e\left(\frac{\partial T}{\partial n}\right)_{e^+}\right], \end{aligned} \tag{5.17}$$

and $\lambda_\mathrm{H} \equiv \int_0^\infty \mathrm{d}\tau\, \frac{1}{3}\langle \boldsymbol{J}_\mathrm{E}^D(\tau)\cdot \boldsymbol{J}_\mathrm{E}^D\rangle$.

The general eigenvalues of (5.11) are given by complicated expressions involving cube roots and suggest nonanalytic k and λ behavior from which it is difficult to gain any insight, particularly since there are new transport coefficients involved, and are given elsewhere [9]. We therefore consider the limiting case in which $\lambda \sim k^2$.

When $\lambda \sim k^2$ we find that the hydrodynamic modes are unchanged from the

$\lambda = 0$ case given in eq. (5.14) to lowest order in k. Furthermore, the real part of the hydrodynamic roots are shifted from Γ_{T} to $\tilde{\Gamma}_{\mathrm{T}}$ and from Γ_{S} to $\tilde{\Gamma}_{\mathrm{S}}$, where

$$\tilde{\Gamma}_{\mathrm{T}} = \Gamma_{\mathrm{T}} + \frac{1}{mc_0^2}\frac{\lambda}{k^2}\left[\chi_n\left(\frac{\partial T}{\partial e^+}\right)_n - \chi_e\left(\frac{\partial T}{\partial n}\right)_{e^+}\right]\frac{\Lambda_1}{mT},$$

$$\tilde{\Gamma}_{\mathrm{S}} = \frac{1}{2}\left[\lambda_{\mathrm{H}}\left(\frac{\partial T}{\partial e^+}\right)_n + \frac{\zeta + \frac{4}{3}\eta}{mn_0} - \tilde{\Gamma}_{\mathrm{T}}\right]. \tag{5.18}$$

6. Steady states

There exists a steady state for the granular flow system in which

$$\boldsymbol{v}(\boldsymbol{r}, t) = by\hat{\boldsymbol{e}}_x, \tag{6.1}$$

where b is small and the pressure P_{h} (or internal energy density e^+) and density n are independent of position. In this steady state, all the thermodynamic and transport coefficients are independent of $\boldsymbol{r}$, and the steady-state form of the nonlinear energy equation yields the condition

$$b^2\eta = \left(\frac{T_{\mathrm{H}}}{T} - 1\right)\left(\frac{\lambda\Lambda_1}{m} - \frac{\lambda^2\Lambda_2}{K_{\mathrm{B}}T_{\mathrm{H}}}\right), \tag{6.2}$$

where T_{H} and T are the nonequilibrium and equilibrium temperature of the system respectively, and the granular flow transport coefficients Λ_1 and Λ_2 are given by

$$V\Lambda_1 = \sum_j \sum_{k \neq j} \langle \gamma_{jk} \rangle_{\mathrm{H}}(t), \qquad V\Lambda_2 = \int_0^\infty \mathrm{d}\tau \langle X^{\mathrm{d}}(\tau) Y \rangle_{\mathrm{H}}(t), \tag{6.3}$$

where

$$X = \sum_j \sum_{k \neq j} \frac{\gamma_{jk}}{m}\left(\frac{\beta}{2m}\hat{\boldsymbol{r}}_{jk}\hat{\boldsymbol{r}}_{jk} : (\boldsymbol{p}_j^\dagger - \boldsymbol{p}_k^\dagger)(\boldsymbol{p}_j^\dagger - \boldsymbol{p}_k^\dagger) - 1\right),$$

$$Y = \sum_j \sum_{k \neq j} \frac{\gamma_{jk}}{m}\left(\frac{\beta_{\mathrm{H}}}{2m}\hat{\boldsymbol{r}}_{jk}\hat{\boldsymbol{r}}_{jk} : (\boldsymbol{p}_j^\dagger - \boldsymbol{p}_k^\dagger)(\boldsymbol{p}_j^\dagger - \boldsymbol{p}_k^\dagger) - 1\right). \tag{6.4}$$

The steady-state form of eq. (4.23) to linear order in $\boldsymbol{\phi}^{(2)}$ is

$$\frac{\partial \overline{Q_2}(t)}{\partial t} = 0 = \langle O^{\dagger} Q_2 \rangle_{\mathrm{H}}^{\mathrm{ss}} + \int_0^{\infty} \mathrm{d}\tau \, \langle (\mathcal{Q}_{\mathrm{L}} O^{\dagger} Q_2)(\tau) \, \psi_{\mathrm{L}} \rangle_{\mathrm{H}}^{\mathrm{ss}}$$
$$+ \left(\langle \hat{A}_{\mathrm{L}} (\hat{A}_{\mathrm{L}})_i \rangle_{\mathrm{H}}(t) \frac{\delta \dot{a}(t)}{\delta \boldsymbol{\phi}_j^{(1)}(t)} + \frac{\delta \dot{a}(t)}{\delta \boldsymbol{\phi}_i^{(1)}(t)} \langle \hat{A}_{\mathrm{L}} (\hat{A}_{\mathrm{L}})_j \rangle_{\mathrm{H}}(t) \right)^{\mathrm{ss}} \boldsymbol{\phi}_{ij}^{(2)\mathrm{ss}} , \quad (6.5)$$

where all correlation functions are evaluated in the steady-state local equilibrium ensemble, which in the case of Couette flow, is characterized by

$$\boldsymbol{\phi}_N^{(1)\mathrm{ss}} = -\tfrac{1}{2} \beta_{\mathrm{H}} b^2 y^2 + \beta_{\mathrm{H}} \mu_{\mathrm{H}} - \beta \mu , \qquad \boldsymbol{\phi}_E^{(1)\mathrm{ss}} = \beta - \beta_{\mathrm{H}} ,$$
$$\boldsymbol{\phi}_{\boldsymbol{P}}^{(1)\mathrm{ss}} = \beta_{\mathrm{H}} b y \hat{\boldsymbol{e}}_x . \quad (6.6)$$

We are interested in solving eq. (6.5) for the generalized forces $\boldsymbol{\phi}^{(2)}$ in the regime where b is small. When b is small and e^+ and n are uniform, $\dot{a}$ may be linearized and the linear granular flow hydrodynamic equations analyzed in section 5 are obtained. Therefore, in the regime where b is small, the simplest way to proceed to solve for the $\boldsymbol{\phi}^{(2)}$ is to define the basis set A to be composed of the eigenmodes of the linearized equation for $\dot{a}$. For granular flow systems in which the small parameter λ characterizing the rate of energy loss into the internal modes of the granular particles is on the order of the magnitude of the gradients of the system squared ($\lambda \sim k^2$), we found in section 5 that the hydrodynamic modes to leading order in k are

$$T(k) = \hat{E}^+(k) - \frac{h^+}{n_0} \hat{N}(k) ,$$
$$S^+(k) = \frac{\chi_n}{c_0} \hat{N}(k) + \frac{\chi_e}{c_0} \hat{E}^+(k) + \hat{\boldsymbol{k}} \cdot \boldsymbol{P}^+(k) ,$$
$$S^-(k) = \frac{\chi_n}{c_0} \hat{N}(k) + \frac{\chi_e}{c_0} \hat{E}^+(k) - \hat{\boldsymbol{k}} \cdot \boldsymbol{P}^+(k) ,$$
$$\eta_i(k) = [\hat{\boldsymbol{k}} \wedge \boldsymbol{P}^+(k)]_i , \quad (6.7)$$

where c_0 is the zero frequency adiabatic sound velocity defined by

$$c_0^2 = \frac{\chi_n}{m} + \frac{\chi_e h^+}{m n_0} . \quad (6.8)$$

These modes have the corresponding eigenvalues

$$S^+(k) \rightarrow \mathrm{i} c_0 k - \tilde{\Gamma}_{\mathrm{S}} k^2 , \qquad S^-(k) \rightarrow -\mathrm{i} c_0 k - \tilde{\Gamma}_{\mathrm{S}} k^2 ,$$
$$T(k) \rightarrow -\tilde{\Gamma}_{\mathrm{T}} k^2 , \qquad \eta_i(k) \rightarrow -\nu k^2 , \quad (6.9)$$

where $i = 1, 2$,

$$\tilde{\Gamma}_{\mathrm{T}} = \Gamma_{\mathrm{T}} + \frac{c^2}{c_0^2}\frac{\lambda}{k^2}\left(\frac{\partial T}{\partial e^+}\right)_n \frac{A_1}{mT}\,,$$

$$\tilde{\Gamma}_{\mathrm{S}} = \Gamma_{\mathrm{S}} - \frac{c^2}{c_0^2}\frac{\lambda}{k^2}\left(\frac{\partial T}{\partial e^+}\right)_n \frac{A_1}{mT}\,,$$

$$\nu = \frac{\eta}{mn_0}\,,$$

$$\left(\frac{\partial T}{\partial e^+}\right)_n c^2 = \frac{\chi_n}{m}\left(\frac{\partial T}{\partial e^+}\right)_n - \frac{\chi_e}{m}\left(\frac{\partial T}{\partial n}\right)_{e^+}\,, \tag{6.10}$$

η is the shear viscosity and Γ_{T} and Γ_{S} are the hydrodynamic thermal and sound attenuation coefficients respectively.

The leading order contribution to $\boldsymbol{\phi}^{(2)}$ in the small parameters b and $k \gg \lambda$ comes from the Euler term $\langle O^{\dagger} Q_2 \rangle_{\mathrm{H}}^{\mathrm{ss}}$ since $\psi_{\mathrm{L}}(t)$ has additional small parameters associated with it. If we define the set A to be composed of S^+, S^-, T and η_i, for

$$\begin{aligned} Q_2(\tfrac{1}{2}\boldsymbol{k} + \boldsymbol{q}, \tfrac{1}{2}\boldsymbol{k} - \boldsymbol{q}) &= \hat{A}_{\mathrm{L}}(\boldsymbol{q} + \tfrac{1}{2}\boldsymbol{k})\, \hat{A}_{\mathrm{L}}(\boldsymbol{q} - \tfrac{1}{2}\boldsymbol{k})^* \\ &\quad - \langle \hat{A}_{\mathrm{L}}(\boldsymbol{q} + \tfrac{1}{2}\boldsymbol{k})\, \hat{A}_{\mathrm{L}}(\boldsymbol{q} - \tfrac{1}{2}\boldsymbol{k})^* \rangle_{\mathrm{L}} \\ &\quad - \langle \hat{A}_{\mathrm{L}}(\boldsymbol{q} + \tfrac{1}{2}\boldsymbol{k})\, \hat{A}_{\mathrm{L}}(\boldsymbol{q} - \tfrac{1}{2}\boldsymbol{k})^* \hat{A}_{\mathrm{L}}(\boldsymbol{k})^* \rangle_{\mathrm{L}} \cdot K_{11}^{(\mathrm{L})-1} \cdot \hat{A}_{\mathrm{L}}(\boldsymbol{k})\,, \end{aligned}$$

we obtain to leading order in the small parameters $\langle O^{\dagger} Q_{T\eta_i} \rangle_{\mathrm{L}} = 0$, and

$$\begin{aligned} \langle O^{\dagger} Q_{S^+S^+} \rangle_{\mathrm{L}} &= -\hat{\boldsymbol{q}} \cdot \boldsymbol{S} \cdot \hat{\boldsymbol{q}} + a\left(\frac{T_{\mathrm{H}}}{T} - 1\right) = \langle O^{\dagger} Q_{S^-S^-} \rangle_{\mathrm{L}}\,, \\ \langle O^{\dagger} Q_{S^+S^-} \rangle_{\mathrm{L}} &= \hat{\boldsymbol{q}} \cdot \boldsymbol{S} \cdot \hat{\boldsymbol{q}} + g\left(\frac{T_{\mathrm{H}}}{T} - 1\right) = \langle O^{\dagger} Q_{S^-S^+} \rangle_{\mathrm{L}}\,, \\ \langle O^{\dagger} Q_{TT} \rangle_{\mathrm{L}} &= d\left(\frac{T_{\mathrm{H}}}{T} - 1\right), \\ \langle O^{\dagger} Q_{TS^+} \rangle_{\mathrm{L}} &= e\left(\frac{T_{\mathrm{H}}}{T} - 1\right) = \langle O^{\dagger} Q_{TS^-} \rangle_{\mathrm{L}}\,, \\ \langle O^{\dagger} Q_{\eta_i\eta_i} \rangle_{\mathrm{L}} &= -\hat{\boldsymbol{q}} \wedge (\hat{\boldsymbol{q}} \wedge \boldsymbol{S} \wedge \hat{\boldsymbol{q}}) \wedge \hat{\boldsymbol{q}} + (\boldsymbol{I} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) f\left(\frac{T_{\mathrm{H}}}{T} - 1\right), \\ \langle O^{\dagger} Q_{S^+\eta_i} \rangle_{\mathrm{L}} &= -(\hat{\boldsymbol{q}} \cdot \boldsymbol{S} \wedge \hat{\boldsymbol{q}}) \wedge \hat{\boldsymbol{q}}\,, \end{aligned} \tag{6.11}$$

where $\boldsymbol{S}$ is the shear tensor

$$\mathbf{S}_{ij} \equiv \frac{1}{2}\left(\frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{r}_j} + \frac{\partial \boldsymbol{v}_j}{\partial \boldsymbol{r}_i}\right)$$

and a, g, d, e and f are thermodynamic functions of the system given by

$$\begin{aligned}
a &= \frac{2\Lambda_1}{\beta_{\mathrm{H}}}\left\{\frac{\chi_e}{mc_0^2}\left[1 + \frac{n_0}{c_0}\left(\frac{\partial c_0}{\partial n}\right)_{\tilde{s}}\right]\right. \\
&\quad \left. + \frac{n_0}{C_{\mathrm{p}}T_{\mathrm{H}}}\left[\frac{T_{\mathrm{H}}}{n_0}\left(\frac{\partial n}{\partial T}\right)_P + \frac{T_{\mathrm{H}}}{c_0}\left(\frac{\partial c_0}{\partial T}\right)_P + 1\right]\right\} - \frac{2n_0\chi_e}{mc_0^2\beta_{\mathrm{H}}}\left(\frac{\partial \Lambda_1}{\partial n}\right)_{\tilde{s}} , \\
g &= \frac{2\Lambda_1}{\beta_{\mathrm{H}}}\left\{\frac{\chi_e}{mc_0^2}\left[\frac{n_0}{c_0}\left(\frac{\partial c_0}{\partial n}\right)_{\tilde{s}} - \chi_e\right] + \frac{n_0}{C_{\mathrm{p}}T_{\mathrm{H}}}\frac{T_{\mathrm{H}}}{c_0}\left(\frac{\partial c_0}{\partial T}\right)_P\right\} - \frac{2\chi_e n_0}{mc_0^2\beta_{\mathrm{H}}}\left(\frac{\partial \Lambda_1}{\partial n}\right)_{\tilde{s}} , \\
d &= \frac{\Lambda_1}{m\beta_{\mathrm{H}}}\left\{\frac{\chi_e C_{\mathrm{p}} T_{\mathrm{H}}}{mn_0c_0^2}\left[2\chi_e + \frac{n_0}{C_{\mathrm{p}}}\left(\frac{\partial C_{\mathrm{p}}}{\partial n}\right)_{\tilde{s}}\right] + \frac{T_{\mathrm{H}}}{C_{\mathrm{p}}}\left(\frac{\partial C_{\mathrm{p}}}{\partial T}\right)_P\right\} - \frac{2T_{\mathrm{H}}}{m\beta_{\mathrm{H}}}\left(\frac{\partial \Lambda_1}{\partial T}\right)_P , \\
e &= \frac{\Lambda_1}{\beta_{\mathrm{H}}mc_0}\left[\chi_e + \frac{n_0}{C_{\mathrm{p}}}\left(\frac{\partial C_{\mathrm{p}}}{\partial n}\right)_{\tilde{s}} + \frac{\chi_e T_{\mathrm{H}}}{n_0}\left(\frac{\partial n}{\partial T}\right)_P + \frac{2\chi_e T_{\mathrm{H}}}{c_0}\left(\frac{\partial c_0}{\partial T}\right)_P\right] \\
&\quad - \frac{1}{mc_0}\left[\frac{n_0}{\beta_{\mathrm{H}}}\left(\frac{\partial \Lambda_1}{\partial n}\right)_{\tilde{s}} + \frac{\chi_e T_{\mathrm{H}}}{\beta_{\mathrm{H}}}\left(\frac{\partial \Lambda_1}{\partial T}\right)_P\right] , \\
f &= \frac{\Lambda_1}{\beta_{\mathrm{H}}}\left\{\frac{\chi_e}{mc_0^2}(1 + \chi_e) + \frac{n_0}{C_{\mathrm{p}}T_{\mathrm{H}}}\left[\frac{T_{\mathrm{H}}}{n_0}\left(\frac{\partial n}{\partial T}\right)_P + 1\right]\right\} ,
\end{aligned} \tag{6.12}$$

where $V\Lambda_1 \equiv \langle \Sigma_j \, \Sigma_{k \neq j} \, \gamma_{jk} \rangle_{\mathrm{H}}$ and $\tilde{S} \equiv S/N$ is the entropy per particle. To evaluate eq. (6.11), we have used the facts that

$$\begin{aligned}
\langle \hat{B}T \rangle_{\mathrm{H}} &= \frac{T_{\mathrm{H}}V}{\beta_{\mathrm{H}}}\left(\frac{\partial b}{\partial T}\right)_P , & \langle \hat{B}\Pi_0 \rangle_{\mathrm{H}} &= \frac{n_0V}{\beta_{\mathrm{H}}}\left(\frac{\partial b}{\partial n}\right)_{\tilde{s}} , \\
\left(\frac{\partial T}{\partial n}\right)_{\tilde{s}} &= \frac{T_{\mathrm{H}}}{n_0}\chi_e , & \left(\frac{\partial e}{\partial n}\right)_{\tilde{s}} &= \frac{h^+}{n_0} ,
\end{aligned} \tag{6.13}$$

where $\Pi_0 \equiv \chi_n \hat{N} + \chi_e \hat{E}$ and $T \equiv \hat{E} - (h^+/n_0)\hat{N}$. We have also used the identity for arbitrary vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$,

$$\boldsymbol{x} \wedge (\boldsymbol{y} \wedge \boldsymbol{z}) = (\boldsymbol{x} \cdot \boldsymbol{z})\boldsymbol{y} - (\boldsymbol{x} \cdot \boldsymbol{y})\boldsymbol{z} , \tag{6.14}$$

which allows us to write

$$\boldsymbol{P} = \hat{\boldsymbol{q}}(\hat{\boldsymbol{q}} \cdot \boldsymbol{P}) + \hat{\boldsymbol{q}} \wedge (\hat{\boldsymbol{q}} \wedge \boldsymbol{P}) . \tag{6.15}$$

Using eq. (6.11), we can now solve (6.5) for $\boldsymbol{\phi}^{(2)}$ to leading order in the small parameters, k and q, to obtain $\boldsymbol{\phi}^{(2)}_{T\eta_i} = 0$ and

$$
\begin{aligned}
\phi^{(2)}_{S^+S^+}(\boldsymbol{k}, \boldsymbol{q}) &= \frac{-(\hat{\boldsymbol{q}} \cdot \boldsymbol{S} \cdot \hat{\boldsymbol{q}})}{4mn_0K_\mathrm{B}T_\mathrm{H}\tilde{\Gamma}_\mathrm{S}q^2} + \left(\frac{T_\mathrm{H}}{T} - 1\right)\frac{a}{4(mn_0K_\mathrm{B}T_\mathrm{H})^2\tilde{\Gamma}_\mathrm{S}q^2} \\
&= \boldsymbol{\phi}^{(2)}_{S^-S^-}(\boldsymbol{k}, \boldsymbol{q}) , \\
\phi^{(2)}_{S^+S^-}(\boldsymbol{k}, \boldsymbol{q}) &= \frac{-\mathrm{i}(\hat{\boldsymbol{q}} \cdot \boldsymbol{S} \cdot \hat{\boldsymbol{q}})}{4mn_0K_\mathrm{B}T_\mathrm{H}c_0q} + \left(\frac{T_\mathrm{H}}{T} - 1\right)\frac{-\mathrm{i}g}{4(mn_0K_\mathrm{B}T_\mathrm{H})^2c_0q} \\
&= -\boldsymbol{\phi}^{(2)}_{S^-S^+}(\boldsymbol{k}, \boldsymbol{q}) , \\
\phi^{(2)}_{TT}(\boldsymbol{k}, \boldsymbol{q}) &= \left(\frac{T_\mathrm{H}}{T} - 1\right)\frac{d}{(C_\mathrm{p}T_\mathrm{H}K_\mathrm{B}T_\mathrm{H})^2\tilde{\Gamma}_\mathrm{T}q^2} , \\
\phi^{(2)}_{TS^+}(\boldsymbol{k}, \boldsymbol{q}) &= \left(\frac{T_\mathrm{H}}{T} - 1\right)\frac{\mathrm{i}e}{4C_\mathrm{p}T_\mathrm{H}(K_\mathrm{B}T_\mathrm{H})^2mn_0c_0q} = -\boldsymbol{\phi}^{(2)}_{TS^-}(\boldsymbol{k}, \boldsymbol{q}) , \\
\phi^{(2)}_{\eta\eta}(\boldsymbol{k}, \boldsymbol{q}) &= \frac{-\hat{\boldsymbol{q}} \wedge (\hat{\boldsymbol{q}} \wedge \boldsymbol{S} \wedge \hat{\boldsymbol{q}}) \wedge \hat{\boldsymbol{q}}}{mn_0K_\mathrm{B}T_\mathrm{H}\nu q^2} + \left(\frac{T_\mathrm{H}}{T} - 1\right)\frac{f(\boldsymbol{I} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}})}{2(mn_0K_\mathrm{B}T_\mathrm{H})^2\nu q^2} , \\
\phi^{(2)}_{S^+\eta}(\boldsymbol{k}, \boldsymbol{q}) &= \frac{\mathrm{i}(\hat{\boldsymbol{q}} \cdot \boldsymbol{S} \wedge \hat{\boldsymbol{q}}) \wedge \hat{\boldsymbol{q}}}{2mn_0K_\mathrm{B}T_\mathrm{H}c_0q} = \boldsymbol{\phi}^{(2)}_{S^-\eta}(\boldsymbol{k}, \boldsymbol{q}) ,
\end{aligned}
\tag{6.16}
$$

We define χ_{AA} by

$$
\begin{aligned}
\chi_{AA} &= \overline{Q_2(A(\boldsymbol{q} + \tfrac{1}{2}\boldsymbol{k})A(\boldsymbol{q} - \tfrac{1}{2}\boldsymbol{k})^*)} \\
&= \overline{\hat{A}_\mathrm{L}(\boldsymbol{q} + \tfrac{1}{2}\boldsymbol{k})\, \hat{A}_\mathrm{L}(\boldsymbol{q} - \tfrac{1}{2}\boldsymbol{k})^*} - \langle \hat{A}_\mathrm{L}(\boldsymbol{q} + \tfrac{1}{2}\boldsymbol{k})\, \hat{A}_\mathrm{L}(\boldsymbol{q} - \tfrac{1}{2}\boldsymbol{k})^* \rangle_\mathrm{L} .
\end{aligned}
$$

Since all thermodynamic and transport coefficients are independent of $\boldsymbol{r}$, we may write χ_{AA} as

$$
\begin{aligned}
\chi_{AA}(\boldsymbol{k}, \boldsymbol{q}) &= \frac{1}{V^2} \int \mathrm{d}\boldsymbol{R}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}} \langle \hat{A}_\mathrm{L}\hat{A}_\mathrm{L} \rangle_\mathrm{H}(\boldsymbol{R})\, \langle \hat{A}_\mathrm{L}\hat{A}_\mathrm{L} \rangle_\mathrm{H}(\boldsymbol{R}) : \boldsymbol{\psi}^{(2)}(\boldsymbol{R}, \boldsymbol{q}) + \mathrm{O}(M) \\
&= \frac{\langle \hat{A}_\mathrm{L}\hat{A}_\mathrm{L} \rangle_\mathrm{H}}{V} \frac{\langle \hat{A}_\mathrm{L}\hat{A}_\mathrm{L} \rangle_\mathrm{H}}{V} : \boldsymbol{\phi}^{(2)}(\tfrac{1}{2}\boldsymbol{k} + \boldsymbol{q}, \tfrac{1}{2}\boldsymbol{k} - \boldsymbol{q}) .
\end{aligned}
\tag{6.17}
$$

Using the definitions of the eigenmodes and simple thermodynamic relations, we immediately obtain from eqs. (6.16) and (6.17) the final results

$$
\begin{aligned}
\chi_{NN} &= \left(\frac{n_0}{\beta_{\mathrm{H}} c_0^2}\right)^2 \left[\frac{-(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}\cdot\hat{\boldsymbol{q}})}{2mn_0K_{\mathrm{B}}T_{\mathrm{H}}\tilde{\Gamma}_{\mathrm{S}}q^2} + \left(\frac{T_{\mathrm{H}}}{T}-1\right)\frac{a}{2(mn_0K_{\mathrm{B}}T_{\mathrm{H}})^2\tilde{\Gamma}_{\mathrm{S}}q^2}\right] \\
&\quad + \left[\frac{T_{\mathrm{H}}}{\beta_{\mathrm{H}}}\left(\frac{\partial n}{\partial T}\right)_P\right]^2 \left(\frac{T_{\mathrm{H}}}{T}-1\right)\frac{c}{(C_{\mathrm{p}}T_{\mathrm{H}}K_{\mathrm{B}}T_{\mathrm{H}})^2\tilde{\Gamma}_{\mathrm{T}}q^2}\,, \\
\chi_{NE^+} &= \left(\frac{n_0h^+}{(\beta_{\mathrm{H}} c_0)^2}\right) \left[\frac{-(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}\cdot\hat{\boldsymbol{q}})}{2mn_0K_{\mathrm{B}}T_{\mathrm{H}}\tilde{\Gamma}_{\mathrm{S}}q^2} + \left(\frac{T_{\mathrm{H}}}{T}-1\right)\frac{a}{2(mn_0K_{\mathrm{B}}T_{\mathrm{H}})^2\tilde{\Gamma}_{\mathrm{S}}q^2}\right] \\
&\quad + \left(\frac{T_{\mathrm{H}}}{\beta_{\mathrm{H}}}\right)^2 \left(\frac{\partial e^+}{\partial T}\right)_P \left(\frac{\partial n}{\partial T}\right)_P \left(\frac{T_{\mathrm{H}}}{T}-1\right)\frac{c}{(C_{\mathrm{p}}T_{\mathrm{H}}K_{\mathrm{B}}T_{\mathrm{H}})^2\tilde{\Gamma}_{\mathrm{T}}q^2}\,, \\
\chi_{E^+E^+} &= \left(\frac{h^+}{\beta_{\mathrm{H}} c_0}\right)^2 \left[\frac{-(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}\cdot\hat{\boldsymbol{q}})}{2mn_0K_{\mathrm{B}}T_{\mathrm{H}}\tilde{\Gamma}_{\mathrm{S}}q^2} + \left(\frac{T_{\mathrm{H}}}{T}-1\right)\frac{a}{2(mn_0K_{\mathrm{B}}T_{\mathrm{H}})^2\tilde{\Gamma}_{\mathrm{S}}q^2}\right] \\
&\quad + \left(\frac{C_{\mathrm{p}}T_{\mathrm{H}}}{\beta_{\mathrm{H}}}\right)^2 \left(\frac{T_{\mathrm{H}}}{T}-1\right)\frac{c}{(C_{\mathrm{p}}T_{\mathrm{H}}K_{\mathrm{B}}T_{\mathrm{H}})^2\tilde{\Gamma}_{\mathrm{T}}q^2}\,, \\
\chi_{\boldsymbol{P}^+\boldsymbol{P}^+} &= mn_0K_{\mathrm{B}}T_{\mathrm{H}}\left[\frac{-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}\cdot\hat{\boldsymbol{q}})}{2\tilde{\Gamma}_{\mathrm{S}}q^2} + \left(\frac{T_{\mathrm{H}}}{T}-1\right)\frac{a\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}}{2(mn_0K_{\mathrm{B}}T_{\mathrm{H}})\tilde{\Gamma}_{\mathrm{S}}q^2}\right] \\
&\quad + mn_0K_{\mathrm{B}}T_{\mathrm{H}}\left[\frac{-\hat{\boldsymbol{q}}\wedge(\hat{\boldsymbol{q}}\wedge\boldsymbol{S}\wedge\hat{\boldsymbol{q}})\wedge\hat{\boldsymbol{q}}}{\nu q^2}\right. \\
&\quad \left. + \left(\frac{T_{\mathrm{H}}}{T}-1\right)\frac{f(\boldsymbol{I}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}})}{2(mn_0K_{\mathrm{B}}T_{\mathrm{H}})\nu q^2}\right]
\end{aligned}
\tag{6.18}
$$

and

$$
\begin{aligned}
\chi_{N\boldsymbol{P}^+} &= \frac{\mathrm{i}(mn_0k_{\mathrm{B}}T_{\mathrm{H}})^2}{mc_0^2q}\left[\frac{2(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}\wedge\hat{\boldsymbol{q}})\wedge\hat{\boldsymbol{q}} + (\hat{\boldsymbol{q}}\cdot\boldsymbol{S}\cdot\hat{\boldsymbol{q}})\hat{\boldsymbol{q}}}{2mn_0k_{\mathrm{B}}T_{\mathrm{H}}}\right. \\
&\quad \left. + \left(\frac{T_{\mathrm{H}}}{T}-1\right)\frac{g\hat{\boldsymbol{q}}}{2(mn_0k_{\mathrm{B}}T_{\mathrm{H}})^2}\right] + \frac{2\mathrm{i}e\hat{\boldsymbol{q}}}{c_0q}\left(\frac{T_{\mathrm{H}}}{T}-1\right)\frac{T_{\mathrm{H}}}{C_{\mathrm{p}}T_{\mathrm{H}}}\left(\frac{\partial n}{\partial T}\right)_P\,, \\
\chi_{E^+\boldsymbol{P}^+} &= \frac{\mathrm{i}mn_0h^+(k_{\mathrm{B}}T_{\mathrm{H}})^2}{mc_0^2q}\left[\frac{2(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}\wedge\hat{\boldsymbol{q}})\wedge\hat{\boldsymbol{q}} + (\hat{\boldsymbol{q}}\cdot\boldsymbol{S}\cdot\hat{\boldsymbol{q}})\hat{\boldsymbol{q}}}{2mn_0k_{\mathrm{B}}T_{\mathrm{H}}}\right. \\
&\quad \left. + \left(\frac{T_{\mathrm{H}}}{T}-1\right)\frac{g\hat{\boldsymbol{q}}}{2(mn_0k_{\mathrm{B}}T_{\mathrm{H}})^2}\right] + \left(\frac{T_{\mathrm{H}}}{T}-1\right)\frac{2\mathrm{i}e\hat{\boldsymbol{q}}}{c_0q}\,.
\end{aligned}
\tag{6.19}
$$

Thus we may conclude that even tensored equal-time correlations in granular flow systems have a nonanalytic, $1/q^2$ dependence on the relative wavevector q which implies the correlations are relatively long-ranged. It should be noted, however, that the generalized thermodynamic forces $\boldsymbol{\phi}^{(2)}(\boldsymbol{R}, \boldsymbol{r}_{12})$ decay algebraically with respect to the relative spatial variable $\boldsymbol{r}_{12}$, and hence our initial assumption that $\boldsymbol{\phi}^{(2)}(\boldsymbol{R}, \boldsymbol{r}_{12})$ decays to zero for $|\boldsymbol{r}_{12}|$ large is consistent with the result we obtained. Note that when $T_{\mathrm{H}} = T$, the results for χ_{AA} reduce to those given elsewhere [12,15] for ordinary hydrodynamic systems in which there is no energy flow into internal modes. The steady-state condition for the linear variable requires that

$$\frac{T_{\mathrm{H}}}{T} - 1 = \frac{mb^2\eta}{\lambda A_1}\left[1 + \mathrm{O}\left(\frac{\lambda m K_{\mathrm{B}} T_{\mathrm{H}} A_2}{A_1}\right)\right], \tag{6.20}$$

which allows us to relate the temperature difference $T_{\mathrm{H}} - T$ to parameters of the system. It is also interesting to note that the granular flow equal-time correlations depend on the thermal damping coefficient $\tilde{\Gamma}_{\mathrm{T}}$ as well as $\tilde{\Gamma}_{\mathrm{S}}$.

Acknowledgements

This research was supported in part by the Pittsburgh Energy Technology division of the Department of Energy. This paper is dedicated to Professor Kyozi Kawasaki on the occasion of his retirement from Kyushu University. Kawasaki's pioneering and seminal work on mode coupling provides the foundation for this work.

Appendix A

A.1. Local equilibrium factorizations

In this subsection, we show that the dependence of the local equilibrium cumulants on the center of mass coordinate does not significantly alter the N-ordering analysis of local equilibrium correlation functions even though the center of mass dependence does imply that each local equilibrium cumulant is not accompanied by a wavevector equality, unlike equilibrium cumulants. For example, the expression

$$\langle Q_2(\boldsymbol{k}, \boldsymbol{k}')\, Q_2(\boldsymbol{q}, \boldsymbol{q}')^* \rangle_{\mathrm{L}} * \boldsymbol{\phi}^{(2)}(\boldsymbol{q}, \boldsymbol{q}', t) \tag{A.1}$$

can be written in terms of a factored and an unfactored part. The factored part has terms which look like

$$\frac{1}{V^4}\sum_{q,q'} \langle \hat{A}_{\mathrm{L}}\hat{A}_{\mathrm{L}}\rangle_{\mathrm{H}}(\boldsymbol{k}-\boldsymbol{q},t)\,\langle \hat{A}_{\mathrm{L}}\hat{A}_{\mathrm{L}}\rangle_{\mathrm{H}}(\boldsymbol{k}'-\boldsymbol{q}',t)$$

$$\times \int \mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2 \exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}_1+\mathrm{i}\boldsymbol{q}'\cdot\boldsymbol{r}_2)\,\boldsymbol{\phi}^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2,t)$$

$$\approx \frac{1}{V^2}\int \mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{r} \exp[\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{R}]\,\langle \hat{A}_{\mathrm{L}}\hat{A}_{\mathrm{L}}\rangle_{\mathrm{H}}(\boldsymbol{R},t)$$

$$\times \langle \hat{A}_{\mathrm{L}}\hat{A}_{\mathrm{L}}\rangle_{\mathrm{H}}(\boldsymbol{R},t)\,\boldsymbol{\psi}^{(2)}(\boldsymbol{R},\boldsymbol{r},t)$$

$$\sim \frac{N^2}{V^2}V\xi^3 \sim N\,,$$

whereas the unfactored part looks like

$$\frac{1}{V^3}\sum_{q,q'} \langle Q_2Q_2\rangle_{\mathrm{H}}(\boldsymbol{k}+\boldsymbol{k}'-\boldsymbol{q}-\boldsymbol{q}',t)\int \mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{r}$$

$$\times \exp[\mathrm{i}(\boldsymbol{q}+\boldsymbol{q}')\cdot\boldsymbol{R}]\exp[\tfrac{1}{2}\mathrm{i}(\boldsymbol{q}-\boldsymbol{q}')\cdot\boldsymbol{r}]\,\boldsymbol{\psi}^{(2)}(\boldsymbol{R},\boldsymbol{r},t)$$

$$\approx \frac{1}{V^2}\sum_{q-q'}\int \mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{r} \exp[\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{R}]\,\langle Q_2Q_2\rangle_{\mathrm{H}}(\boldsymbol{R},t)\,\boldsymbol{\psi}^{(2)}(\boldsymbol{R},\boldsymbol{r},t)$$

$$\sim N\left(\frac{M}{N}\right) \sim M\,.$$

Thus we see that even though the factorizations are not accompanied by wavevector equalities, the N orders are identical to the leading order $\boldsymbol{\phi}^{(2)}(t)$ terms, which involve only equilibrium correlation functions. Similarly, we find that

$$\frac{1}{V^4}\sum_{j,j'}\sum_{l,l'} \langle \hat{A}_{\mathrm{L}}(\boldsymbol{k})\,Q_2(j,j')^*Q_2(l,l')^*\rangle_{\mathrm{L}*}^{\;*}\,\boldsymbol{\phi}^{(2)}(j,j',t)\,\phi^{(2)}(l,l',t) \tag{A.2}$$

is approximately

$$\sum_{l+l'}\int \mathrm{d}\boldsymbol{R}\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\,\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{R}}}{V^3}\,\langle \hat{A}_{\mathrm{L}}\hat{A}_{\mathrm{L}}\hat{A}_{\mathrm{L}}\rangle_{\mathrm{H}}(\boldsymbol{R},t)\,\langle \hat{A}_{\mathrm{L}}\hat{A}_{\mathrm{L}}\rangle_{\mathrm{H}}(\boldsymbol{R},t)$$

$$:\boldsymbol{\psi}^{(2)}(\boldsymbol{R},\boldsymbol{r},t)\,\boldsymbol{\psi}^{(2)}(\boldsymbol{R},\boldsymbol{r}',t)$$

$$\sim \frac{N^2}{V}(K_{\mathrm{c}}\xi)^3\xi^3 \sim N\left(\frac{M}{N}\right) = M\,, \tag{A.3}$$

as was observed in eq. (3.31) of the main text. In general, it is found that the additional sums present in the factorization of local equilibrium correlation functions which do not exist in equilibrium correlations due to translational invariance do not change the overall N orders.

A.2. Position space factorizations

In this subsection, we analyze the approximation scheme used in section 3 by examining correlations in position space rather than wavevector space. Due to the short-ranged nature of the local equilibrium distribution function, we expect that for $r_{12} = |\boldsymbol{r}_{12}|$ larger than a short correlation length a, the correlation function $\langle Q_2(\boldsymbol{r}_1, \boldsymbol{r}_2)\, Q_2(\boldsymbol{r}_3, \boldsymbol{r}_4)\rangle_{\mathrm{L}}$ factors such that

$$\begin{aligned}\langle Q_2(\boldsymbol{r}_1, \boldsymbol{r}_2)\, Q_2(\boldsymbol{r}_3, \boldsymbol{r}_4)\rangle_{\mathrm{L}} &= \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_3)\rangle_{\mathrm{L}} \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_2)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_4)\rangle_{\mathrm{L}} \\ &\quad + \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_4)\rangle_{\mathrm{L}} \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_2)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_3)\rangle_{\mathrm{L}} \,.\end{aligned}$$

The correlation function

$$\langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_3)\rangle_{\mathrm{L}} \tag{A.4}$$

vanishes if $|\boldsymbol{r}_1 - \boldsymbol{r}_3| > a$, where a represents the local equilibrium correlation length which we anticipate to be on the order of angstroms and smaller than ξ, the range of the relative arguments of the generalized thermodynamic forces $\boldsymbol{\phi}(t)$. Thus, each average like (A.4), which vanishes when its spatial arguments are separated, is approximately proportional to the function $1 - H(|\boldsymbol{r}_1 - \boldsymbol{r}_3| - a)$, where the Heaviside function $H(r)$ is defined by

$$H(r) = \begin{cases} 1 & \text{if}\, r \geqslant 0\,, \\ 0 & \text{if}\, r < 0\,. \end{cases} \tag{A.5}$$

Since

$$\begin{aligned}\langle Q_2(\boldsymbol{r}_1, \boldsymbol{r}_2)\, Q_2(\boldsymbol{r}_3, \boldsymbol{r}_4)\rangle_{\mathrm{L}}^{\mathrm{unf}} \equiv &\ \langle Q_2(\boldsymbol{r}_1, \boldsymbol{r}_2)\, Q_2(\boldsymbol{r}_3, \boldsymbol{r}_4)\rangle_{\mathrm{L}} \\ &- \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_3)\rangle_{\mathrm{L}} \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_2)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_4)\rangle_{\mathrm{L}} \\ &- \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_1)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_4)\rangle_{\mathrm{L}} \langle \hat{A}_{\mathrm{L}}(\boldsymbol{r}_2)\, \hat{A}_{\mathrm{L}}(\boldsymbol{r}_3)\rangle_{\mathrm{L}}\end{aligned} \tag{A.6}$$

vanishes if the separation between any two of the $\boldsymbol{r}_i$ exceeds a, we expect that $\langle Q_2 Q_2\rangle_{\mathrm{L}}^{\mathrm{unf}}$ is proportional to

$$[1 - H(|\boldsymbol{r}_2 - \boldsymbol{r}_1| - a)][1 - H(|\boldsymbol{r}_3 - \boldsymbol{r}_1| - a)][1 - H(|\boldsymbol{r}_4 - \boldsymbol{r}_1| - a)]\,.$$

Since the matrix $\tilde{M}_{22}(t)$ can be written as

$$\tilde{M}_{22}(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{R}_1,\boldsymbol{R}_2,t)=\tilde{M}_{11}(\boldsymbol{r}_1;\boldsymbol{R}_1,t)\,\delta(\boldsymbol{r}_2-\boldsymbol{R}_2)+\tilde{M}_{11}(\boldsymbol{r}_1;\boldsymbol{R}_2,t)\,\delta(\boldsymbol{r}_2-\boldsymbol{R}_1)$$
$$+\tilde{M}_{11}(\boldsymbol{r}_2;\boldsymbol{R}_1,t)\,\delta(\boldsymbol{r}_1-\boldsymbol{R}_2)+\tilde{M}_{11}(\boldsymbol{r}_2;\boldsymbol{R}_2,t)\,\delta(\boldsymbol{r}_1-\boldsymbol{R}_1)$$
$$+\tilde{M}_{22}^{\mathrm{unf}}(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{R}_1,\boldsymbol{R}_2,t)\,,$$

we see that the term

$$\tilde{M}_{22}(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{R}_1,\boldsymbol{R}_2,t)*\overline{Q_2(\boldsymbol{R}_1,\boldsymbol{R}_2,t)}^{0}$$

in the equation for $\overline{Q_2(\boldsymbol{R}_1,\boldsymbol{R}_2,t)}^{0}$ has terms like

$$\tilde{M}_{11}(\boldsymbol{r}_1;\boldsymbol{R}_1,t)\cdot\overline{Q_2(\boldsymbol{R}_1,\boldsymbol{r}_2,t)}^{0}\,, \tag{A.7}$$

$$\tilde{M}_{22}^{\mathrm{unf}}(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{R}_1,\boldsymbol{R}_2,t)*\overline{Q_2(\boldsymbol{R}_1,\boldsymbol{R}_2,t)}^{0}\,. \tag{A.8}$$

Now, since $\tilde{M}_{11}(\boldsymbol{r}_1;\boldsymbol{R}_1,t)$ is proportional to $1-H(|\boldsymbol{r}_1-\boldsymbol{R}_1|-a)$ and $\tilde{M}_{22}^{\mathrm{unf}}(t)$ is proportional to

$$[1-H(|\boldsymbol{R}_1-\boldsymbol{r}_1|-a)][1-H(|\boldsymbol{R}_2-\boldsymbol{r}_1|-a)][1-H(|\boldsymbol{r}_2-\boldsymbol{r}_2|-a)]\,,$$

we see that

$$\tilde{M}_{11}(\boldsymbol{r}_1;\boldsymbol{R}_1,t)\cdot\overline{Q_2(\boldsymbol{R}_1,\boldsymbol{r}_2,t)}^{0}\approx\tilde{M}_{11}(\boldsymbol{r}_1,t)\cdot\overline{Q_2(\boldsymbol{r}_1,\boldsymbol{r}_2,t)}^{0}\,,$$

whereas eq. (A.8) gives no contribution when $r_{12}>a$. Since we are interested only in the large distance ($r_{12}>a$ or small q) behavior of the equal-time correlations we may neglect (A.8). This argument can be generalized to show that $\tilde{M}_{2m}(t)*\overline{Q_m(t)}^{0}$ does not contribute to the dynamics of $\overline{Q_2(t)}^{0}$ for $|m|\geqslant 3$ when $r_{12}\gg a$, since even if $\tilde{M}_{2m}(t)$ is factored into $\tilde{M}_{1m-1}(t)$ at least two of the arguments of $\overline{Q_m(t)}^{0}$ are closely grouped within a distance a of one another. Such terms involve nonhydrodynamic equal-time correlations. From these arguments, we finally obtain eq. (3.103) of section 3.

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