Appendix A

Math Appendices

A.1 Taylor expansion

- Expand function $f(x+a)$ from small a around $a=0$.

$$\begin{aligned} f(x+a) &= f(x) + f'(x)a + \frac{1}{2}f''(x)a^2 + \cdots \\ &= \sum_{j=0}^{\infty} \frac{a^j}{j!}\frac{d^j}{dx^j}f(x). \end{aligned}$$

- Since

$$\begin{aligned} e^{\lambda x} &= \exp(\lambda x) = \sum_{j=0}^{\infty} x^j \frac{\lambda^j}{j!}, \\ f(x+a) &= \exp\left(a\frac{d}{dx}\right)f(x). \end{aligned}$$

A.2 Series expansions

For $|x| < 1$,

$$\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots \\
\frac{1}{1-x} &= 1 + x + x^2 + \cdots \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \\
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots
\end{aligned}$$

A.3 Probability theory:

A.3.1 Discrete systems

Suppose have measurable E with n discrete values $E_1, E_2, \ldots, E_n$. Let

$$\begin{aligned}
N &= \text{number of measurements} \\
N_i &= \text{number of measurements of } E_i.
\end{aligned}$$

Then

$$P_i = \text{Probability that } E_i \text{ is measured} = \lim_{N\to\infty} \frac{N_i}{N} \equiv P(E_i)$$

Properties:

1. $0 \leq P_i \leq 1$
2. $\sum_{i=1}^{n} P_i = 1$

Averages:

$$\begin{aligned}
\overline{E} &= \sum_{i=1}^{n} E_i P_i \\
\overline{E^2} &= \sum_{i=1}^{n} E_i^2 P_i \\
\overline{H(E)} &= \sum_{i=1}^{n} H(E_i) P_i
\end{aligned}$$

Variance of E:

$$\begin{aligned}\sigma_E^2 &\equiv \overline{E^2} - \left(\overline{E}\right)^2 \\ &= \overline{\left(E_i - \overline{E}\right)^2}\end{aligned}$$

- σ_E^2 measures the dispersion of the probability distribution: how spread out values are.
- In general, $\sigma_E^2 \neq 0$ unless $P_i = \delta_{ij}$ for some j. This notation means:

$$P_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad \text{which implies } \overline{E} = E_i.$$

- Tchebycheff Inequality:

$$Prob\left(\left|E - \overline{E}\right| \geq \lambda\overline{E}\right) \leq \frac{\sigma_E^2}{\lambda^2\overline{E}^2}.$$

- Joint probability: Suppose N measurements of two properties E and G.

$$\begin{aligned}n_{ij} &= \text{number of measurements of } E_i \text{ and } G_j \\ P_{ij} &= \lim_{N\to\infty} \frac{n_{ij}}{N} \equiv P(E_i, G_j) \equiv \text{joint probability.}\end{aligned}$$

Properties:

1. $\sum_{i.j} P(E_i, G_j) = 1$.
2. $\sum_i P(E_i, G_j) = P(G_j)$.
3. $\sum_j P(E_i, G_j) = P(E_i)$.
4. If E_i and G_j are independent, then $P(E_i, G_j) = P(E_i)P(G_j)$.

A.3.2 Continuous Systems

- Probability of measure an observable X with values between $x, x + dx$ is $p(x)dx$. $p(x)$ is called the "probability density".
  Properties:
  1. Positive definite: $p(x) \geq 0$.
  2. Normalized: $\int_{-\infty}^{\infty} dx p(x) = 1$
- Averages:

$$\begin{aligned}\overline{x} &= \int_{-\infty}^{\infty} dx\, xp(x) \qquad \overline{f(x)} = \int_{-\infty}^{\infty} dx\, f(x)p(x) \\ \sigma_x^2 &= \overline{x^2} - \overline{x}^2 = \int_{-\infty}^{\infty} dx\, \left(x^2 - \overline{x}^2\right) p(x)\end{aligned}$$

A.3.3 Gaussian distributions

1. Distribution specified by first + second moments:

$$P(x) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \mathrm{e}^{-\frac{(x-\langle x\rangle)^2}{2\sigma^2}} \qquad \begin{aligned} \langle x\rangle &= \text{average of x} \\ \langle (x-\langle x\rangle)^2\rangle &= \sigma^2 \end{aligned}$$

2. Important properties (assuming $\langle x\rangle = 0$): $\langle x^{2n+1}\rangle = 0$ and $\langle x^{2n}\rangle = f(\sigma^2)$.

3. If $P(x_1,\ldots,x_n) = \left(\frac{1}{2\pi\langle x_1^2\rangle}\right)^{1/2} \ldots \left(\frac{1}{2\pi\langle x_n^2\rangle}\right)^{1/2} \exp\left\{-\left(\frac{x_1^2}{2\langle x_1^2\rangle}, \ldots + \frac{x_n^2}{2\langle x_n^2\rangle}\right)\right\}$

   Then

$$\begin{aligned} \langle x_i\rangle &= 0 \\ \langle x_i x_j\rangle &= \sigma_i^2\delta_{i,j} \end{aligned}$$

- What happens when $\sigma_x^2 \to 0$? Infinitely narrow distribution, called a *dirac delta function*. Probability density has all the weight on one value.
- There are other representations of the dirac delta function: basically defined in such a way that *one* value receives all the weight.
- Delta functions: defined in a limiting sense.

$$\delta^{(\epsilon)}(x) = \begin{cases} \frac{1}{\epsilon} & -\frac{\epsilon}{2} \le x \le \frac{\epsilon}{2} \\ 0 & |x| > \frac{\epsilon}{2} \end{cases} \qquad \int_{-\infty}^{\infty} dx\ \delta^{(\epsilon)}(x) = \int_{-\epsilon/2}^{\epsilon/2} dx\frac{1}{\epsilon} = 1.$$

$$\int_{-\infty}^{\infty} dx\ \delta^{(\epsilon)}(x) f(x) \approx f(0)\int_{-\infty}^{\infty} dx\ \delta^{(\epsilon)}(x) = f(0) \qquad \text{if } \epsilon \ll 1.$$

  - Function $f(x)$ essentially constant over infinitesimal interval.
  - Definition of delta function: $\delta(x) = \lim_{\epsilon\to 0}\delta^{(\epsilon)}(x)$.

- Representations of delta function in limit $\epsilon \to 0$:
  1. $\frac{1}{2\epsilon}e^{-|x|/\epsilon}$
  2. $\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}$
  3. $\frac{1}{\epsilon\sqrt{\pi}}e^{-x^2/\epsilon^2}$

4. $\frac{1}{\pi}\frac{\sin x/\epsilon}{x}$

– For any continuous function f of x, for all forms above we get

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dx\ \delta^{(\epsilon)}(x-x_0)f(x) = f(x_0).$$

Some properties of the delta function

1. $\delta(-x) = \delta(x)$
2. $\delta(cx) = \frac{1}{|c|}\delta(x)$
3. $\delta[g(x)] = \sum_j \frac{\delta(x-x_j)}{|g'(x_j)|}$ where $g(x_j) = 0$ and $g'(x_j) \neq 0$.
4. $g(x)\delta(x-x_0) = g(x_0)\delta(x-x_0)$
5. $\int_{-\infty}^{\infty} dx\ \delta(x-y)\delta(x-z) = \delta(y-z)$
6. $\int_{-\infty}^{\infty} dx\ \frac{d\delta(x-x_0)}{dx} f(x) = -\int_{-\infty}^{\infty} dx\ \delta(x-x_0)f'(x) = -f'(x_0)$

A.4 Fourier and Laplace Transforms

- *Fourier Transform:*

$$\begin{aligned}\tilde{f}(k) &= \int_{-\infty}^{\infty} e^{ikx} f(x)\ dx \\ f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\tilde{f}(k)\ dk\end{aligned}$$

– Properties:

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-ix(k-k_0)}dx &= 2\pi\delta(k-k_0). \\ \int_{-\infty}^{\infty} f(x-y)g(y)\ dy &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\tilde{f}(k)\tilde{g}(k)\ dk. \\ f(x) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} e^{-x^2/2\sigma^2} &\longrightarrow \tilde{f}(k) = \left(\frac{\sigma^2}{2\pi}\right)^{1/2} e^{-\sigma^2 k^2/2} \\ f(x) = \frac{e^{-\lambda x}}{x} &\longrightarrow \tilde{f}(k) = \frac{4\pi}{k^2+\lambda^2} \\ f(x) = \frac{1}{x} &\longrightarrow \tilde{f}(k) = \frac{4\pi}{k^2}\end{aligned}$$

- *Laplace Transform:*

$$\tilde{f}(z) \;=\; \int_0^{-\infty} e^{-zt} f(t)\, dt$$

  - Useful properties and transforms:

$$\begin{aligned}
\int_0^t f(t-\tau)g(\tau) &= \tilde{f}(z)\tilde{g}(z) \\
f(t) = e^{-at} \;\rightarrow\;& \tilde{f}(z) = \frac{1}{z+a} \\
f(t) = \frac{1}{t^n} \;\rightarrow\;& \tilde{f}(z) = \frac{t^{n-1}}{(n-1)!}.
\end{aligned}$$

A.5 Calculus

A.5.1 Integration by parts

$$\int udv = uv - \int vdu$$

- Example:

$$\int_a^b dx\; f'(x)g(x) = f(x)g(x)\Big|_a^b - \int_a^b dx\; f(x)g'(x)$$

A.5.2 Change of Variable and Jacobians

Let $I = \int_{R_{xy}} dxdy\; f(x,y)$ be the integral over a connected region $R_{x,y}$. Change variables to u, v via the transform $g(u,v) = x$ and $h(u,v) = y$. It follows that:

$$I = \int_{R_{uv}} dudv\; f(g(u,v),h(u,v))\left|\frac{\partial(g,h)}{\partial(u,v)}\right|.$$

where the *Jacobian* $\frac{\partial(g,h)}{\partial(u,v)}$ is

$$\frac{\partial(g,h)}{\partial(u,v)} \equiv \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix}$$

- Example: Suppose

$$I = \int_{-\infty}^{\infty} dxdy\ f(x)g(x-y).$$

Let $u = x$ and $v = x - y$. Under this transformation, range of v is $(-\infty, \infty)$ at a fixed value of u (or x). The Jacobian J is

$$J = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1$$

Thus,

$$I = \int_{-\infty}^{\infty} dudv\ f(u)g(v)|-1| = \int_{-\infty}^{\infty} duf(u)\ \int_{-\infty}^{\infty} dvg(v).$$